

**O‘ZBEKISTON RESPUBLIKASI
OLIV VA O‘RTA MAXSUS TA‘LIM VAZIRLIGI**

URGANCH VA SAMARQAND DAVLAT UNIVERSITETLARI

A.B. Hasanov

**ODDIY DIFFERENSIAL TENGLAMALAR NAZARIYASIGA
KIRISH
(Darslik)**

**5130100 – matematika, 5140300 – mexanika, 5130200 – amaliy
matematika va informatika hamda 5140200-fizika bakalavr
yo‘nalishlarida ta‘lim olayotgan talabalarga mo‘ljallangan**

SAMARQAND – 2019

UDK:517.7.9
BBK:22.161.6
H:31

Hasanov A.B.

Oddiy differensial tenglamalar nazariyasiga kirish
[Matn]/ A.B.Hasanov. – 2019. – 325 b.

Mas'ul muharrir:

Fizika-matematika fanlari doktori, dotsent A.B.Yaxshimuratov

Taqrizchi:

Fizika-matematika fanlari doktori, professor A.Q.O'rinov

Darslik o'zbek, rus va boshqa xorijiy tillarda yozilgan adabiyotlar asosida oliy o'quv yurtlarida 5130100-matematika, 5140300-mexanika, 5130200-amaliy matematika va informatika hamda 5140200-fizika ta'lim yo'nalishlari o'quv rejalari va Davlat ta'lim standartlari talablariga mos ravishda tuzilgan. Darslik oddiy differensial tenglamalar nazariyasining asosiy tushunchalari bilan bir qatorda amaliyotda uchraydigan differensial tenglamalarni yechish usullari batafsil yoritilgan. Bundan tashqari darslikda differensial tenglamalar sistemasi birinchi integrallarining mavjudligi va ularning Gomelton tenglamalar sistemasini integrallashga tatbiqi haqidagi Liuvill teoremasining isboti keltirilgan.

Samarqand – 2019

So‘z boshi

Mazkur darslik Davlat ta‘lim standartlari talablariga mos ravishda o‘zbek, rus va boshqa xorijiy tillarda yozilgan adabiyotlar asosida tuzilgan bo‘lib u oliy o‘quv yurtlarida 5130100-matematika, 5140300-mexanika, 5130200-amaliy matematika va informatika hamda 5140200-fizika yo‘nalishlarida bakalavr tayyorlash o‘quv rejasiga moslashtirib yozilgan. Darslik bob va paragraflarga ajratilgan bo‘lib, unda oddiy differensial tenglamalar nazariyasining asosiy tushunchalari bayon qilingan. Shu bilan bir qatorda amaliyotda uchraydigan differensial tenglamalarning yechish usullari batafsil yoritilgan. Bundan tashqari mazkur darslikda saralangan iqtidorli talabalardan tashkil topgan maxsus guruh a‘zolarini inobatga olgan holda “Oddiy differensial tenglamalar” fanining namunaviy dasturiga qo‘shimcha ravishda ayrim dolzarb mavzular ham kiritilgan. Jumladan, differensial tenglamalar sistemasi birinchi integrallarining mavjudligi va ularning Gomelton tenglamalar sistemasini integrallashga tatbiqi haqidagi Liuvill teoremasining isboti keltirilgan.

Darslikning VII bobida yoritilgan mavzudan talabalar kurs ishlarini hamda bitiruv malakaviy ishlarini yozishda foydalanishlari mumkin.

Ushbu darslik muallifning ko‘p yillik mehnat faoliyati davomida UrDU fizika-matematika va SamDU mexanika-matematika fakultetlarida “Oddiy differensial tenglamalar” fanidan o‘qigan ma‘ruzalari va olib borgan seminar mashg‘ulotlari davomida orttirgan tajribalariga tayangan holda yozilgan. U yetti bobdan iborat bo‘lib, oddiy differensial tenglamalar nazariyasining asosiy tushunchalarini bayon qilishga bag‘ishlangan.

Darslikning I bobi birinchi tartibli oddiy differensial tenglamalar, ularning turlari, ularga qo‘yiladigan Koshi masalalari va yechimga egaligi haqidagi tasdiqlarni isbotlarini bayon qilishga bag‘ishlangan. II bobida xuddi shu masalalar yuqori tartibli differensial tenglamalar uchun bayon qilingan. Bu bobda differensial tenglamalar uchun boshlang‘ich masalalardan tashqari, ikki nuqtali chegaraviy masalalar va ularni yechish usullari ham bayon qilingan. III bobi differensial tenglamalarni qatorlar yordamida integrallashga bag‘ishlangan bo‘lib, unda birinchi tartibli hosilaga nisbatan yechilgan differensial tenglama, birinchi va ikkinchi tartibli chiziqli differensial tenglamalarning golomorf yechimlarini topish usuli bayon qilingan. Misol sifatida Eyri va Bessel tenglamalari keltirilgan.

Darslikning IV bobi oddiy differensial tenglamalar sistemasiga bag‘ishlangan bo‘lib, unda o‘zgarmas koeffitsiyentli chiziqli bir jinsli va bir jinsli bo‘lmagan differensial tenglamalar sistemasining yechish usullari bayon qilingan. Bundan tashqari differensial tenglamalar sistemasi uchun Koshi masalasi umumiy holda bayon qilinib, o‘zgaruvchan koeffitsiyentli chiziqli bir jinsli bo‘lmagan differensial tenglamalar sistemasi uchun Koshi masalasi batafsil o‘rganilgan. V bobi differensial tenglamalarning turg‘unlik

nazariyasiga bag'ishlangan bo'lib, unda turg'unlik tushunchasi ta'riflangan va birinchi tartibli chiziqli differensial tenglamalar sistemasining hamda yuqori tartibli diferensial tenglamalar yechimlarini turg'unlikka tekshirish usullari bayon qilingan. VI bobida differensial tenglamalarning avtonom (muxtor) sistemasi va uning trayektoriyalariga oid ma'lumotlar bayon qilingan.

VII bob avtonom sistemalarning birinchi integrallari va ularning tadbiqlariga bag'ishlangan. Bu bobda birinchi tartibli xususiy hosilali bir jinsli chiziqli va kvazichiziqli tenglamalar va ular uchun Koshi masalasi ham qaralgan.

Kitobdan samarali foydalanadigan o'quvchi matematik analiz va chiziqli algebra fanlarining asosiy tushunchalarini bilishlari zarur deb hisoblayman. Jumladan bu kitob, talabalarni differensial tenglamalarning umumiy nazariyasini va uning tatbiqlarini o'rganishga bo'lgan qiziqishini oshirishga xizmat qilishiga ishonch bildiraman. Mazkur kitobni yozilishida bergan qimmatli maslahatlari uchun prof.A.Q.O'rinovga va kitob matnini terishda bergan yordamlari uchun M.Bekimov hamda O.Mirzayevlarga samimiy minnatdorchilik bildiraman.

Mazkur darslik ayrim kamchilik va xatolardan holi emas. Shu boisdan kitob to'g'risidagi tanqidiy fikr va mulohazalarni mamnuniyat bilan qabul qilaman.

ahasanov2002@mail.ru

Muallif

ODDIY DIFFERENSIAL TENGLAMALAR NAZARIYASIGA KIRISH

Klassik va zamonaviy matematikaning asosiy obyektlaridan biri bu differensial tenglama tushunchasidir, ya'ni erkli o'zgaruvchi, nama'lum funksiya va uning hosilalari orasidagi funksional bog'lanishdir. Nafaqat matematikaning masalalari balki, tabiatda ro'y beradigan bir qator jarayonlarning matematik modeli differensial tenglamaga keltiriladi.

Tabiatda uchraydigan miqdorlarning ko'pchiligi o'zining qonuniga ega. Bu qonunlarni to'g'ridan-to'g'ri topish ancha murakkab masala. Qaralayotgan miqdor, uning o'zgarish tezligi va tezlanishi orasidagi bog'lanishni topish tabiatan ancha yengil. Bu bog'lanishning matematik ifodasi sifatida oddiy differensial tenglamalar hosil bo'ladi.

Jumladan, Nyutonning ikkinchi qonuni quyidagi

$$F = ma$$

tenglama orqali ifodalanadi. Bu yerda F , m masali jismga ta'sir qiluvchi kuch, a esa uning tezlanishi. Aytaylik, m massali jism OX o'q yo'nalishida harakatlansin. U holda uning harakat qonuni $x = x(t)$, a - tezlanish orqali quyidagicha bog'langan:

$$a = \frac{d^2 x(t)}{dt^2}.$$

Agar F - kuch ta'sirida jismning $x(t)$ ko'chishini aniqlash lozim bo'lsa, u ushbu

$$m \frac{d^2 x(t)}{dt^2} = F$$

differensial tenglamaga keltiriladi.

Bundan tashqari, matematik mayatnikning erkin tebranishi tenglamasi:

$$\varphi'' = a \sin \varphi.$$

Bu yerda φ - muvozanat holatdan chetlashish burchagi bo'lib, a - mayatnikning uzunligiga bog'liq bo'lgan o'zgarmas sonidir.

1-ta'rif. Erkli o'zgaruvchi $x \in (a, b)$, noma'lum funksiya $y(x)$ va uning $y'(x)$, $y''(x)$, ..., $y^{(n)}(x)$ hosilalari orasidagi ushbu

$$F(x, y(x), y'(x), \dots, y^{(n)}(x)) = 0 \quad (0.0.1)$$

funksional bog'lanishga n - tartibli oddiy differensial tenglama deyiladi.

2-ta'rif. Tartibi n bo'lgan (0.0.1) tenglamani (a, b) intervalda ayniyatga aylantiruvchi funksiyaga, uning yechimi deyiladi.

Masalan, $y(x) = e^{-kx}$ funksiya quyidagi

$$\frac{dy}{dx} = -ky$$

differensial tenglamaning yechimi ekanligini tekshirish qiyinchilik tug'dirmaydi, ushbu $y(x) = C_1 e^x + C_2 e^{-x} - \sin x$, $C_j = \text{const}$, $j = 1, 2$ ko'rinishdagi funksiyalar quyidagi

$$\frac{d^2 y}{dx^2} - y = \sin x$$

ikkinchi tartibli differensial tenglamaning yechimidan iborat bo'lishini ham osongina ko'rsatish mumkin.

Yuqoridagi, mulohazalardan ixtiyoriy differensial tenglamaning yechimi bor degan fikr kelib chiqmaydi. Masalan ushbu

$$(y')^2 + 1 = 0$$

ko'rinishdagi differensial tenglama yechimga ega emas. Chunki $|y'|^2 \geq 0$.

Differensial tenglama yechimlarining soni bitta yoki cheksiz ko'p bo'lishi mumkin. Masalan ushbu

$$(y'(x))^2 + (y(x))^2 = 0$$

ko'rinishdagi differensial tenglama faqat $y(x) \equiv 0$ nol yechimga ega.

Differensial tenglamalar nazariyasining asosiy masalasi, tenglamaning yechimini topish va topilgan yechimning xossalarini o'rganishdan iborat.

Yechimning x, y o'zgaruvchilar tekisligidagi grafigiga esa (0.0.1) oddiy differensial tenglamaning integral chizig'i deyiladi.

Aytaylik, $y = y(x)$ funksiya ushbu $y' = f(x, y)$, $D(f) = G \subset R^2$ differensial tenglamaning yechimi bo'lsin. U holda $y = y(x)$ funksiyaning grafigi, ya'ni $(x, y(x))$ nuqtalar to'plami G sohada yotuvchi egri chiziqni ifodalaydi. Bu egri chiziqqa $y' = f(x, y)$ differensial tenglamaning integral chizig'i deyiladi.

Oshkormas $\Phi(x, y, c_1, c_2, \dots, c_n) = 0$ funksiya ko'rinishidagi yechimga (0.0.1) tenglamaning integrali deyiladi. Tarkibidagi c_1, c_2, \dots, c_n parametrlarga aniq qiymat berish hisobiga ixtiyoriy yechimni hosil qilish mumkin bo'lsa, bu yechimga (0.0.1) differensial tenglamaning umumiy yechimi deyiladi va $y = \varphi(x, c_1, c_2, \dots, c_n)$ ko'rinishda belgilanadi. Oshkormas $\Phi(x, y, c_1, c_2, \dots, c_n) = 0$ ko'rinishdagi umumiy yechimga (0.0.1) differensial tenglamaning umumiy integrali deyiladi.

Oddiy differensial tenglamalar odatda har xil ko'rinishda bo'lishi mumkin, jumladan

$$\sin y' - y = 0$$

yuqori tartibli hosilaga nisbatan yechilmagan, ikkinchisi esa

$$y'' = \sin y'$$

yuqori tartibli hosilaga nisbatan yechilgan differensial tenglamalar.

3-ta'rif. Yuqori tartibli hosilaga nisbatan yechilgan oddiy differensial tenglamaning umumiy ko'rinishi quyidagicha bo'ladi:

$$y^{(n)}(x) = f(x, y, y', \dots, y^{(n-1)}). \quad (0.0.2)$$

Kelgusida biz, bu turdagi oddiy differensial tenglamaning ushbu

$$y(x_0) = y_0, \quad y'(x_0) = y'_0, \dots, y^{(n-1)}(x_0) = y_0^{(n-1)} \quad (0.0.3)$$

boshlang'ich shartlarni qanoatlantiruvchi yechimini topishga Koshi masalasi deymiz va uning yechimini mavjudligi hamda yagonaligi haqidagi tasdiqlar bilan tanishamiz.

Xususan, hosilaga nisbatan yechilmagan 1-tartibli differensial tenglama

$$F(x, y, y') = 0 \quad (0.0.4)$$

ko'rinishda bo'ladi. Birinchi tartibli hosilaga nisbatan yechilgan differensial tenglama esa

$$y' = f(x, y) \quad (0.0.5)$$

ko'rinishda bo'ladi.

4-ta'rif. Hosilaga nisbatan yechilgan (0.0.5) differensial tenglamaning

$$y(x_0) = y_0 \quad (0.0.6)$$

boshlang'ich shartni qanoatlantiruvchi $y(x)$ yechimini topishga Koshi masalasi deyiladi. Bu yerda x_0 va y_0 oldindan berilgan haqiqiy sonlardir.

Geometrik tilda: $y' = f(x, y)$ tenglamaning (x_0, y_0) nuqtadan o'tuvchi integral chizig'ini topishga Koshi masalasi deyiladi.

Oddiy differensial tenglamalar nazariyasining asosiy masalalaridan biri, bu Koshi masalasi bo'lib, uning yechimi mavjudmi? Agar bunday yechim mavjud bo'lsa, u yagonami? Agar yechim mavjud va yagona bo'lsa, bu yechimni topish algoritmi qanday bo'ladi?, degan savollarga javob berishdan iborat. Bu savollarga javob beradigan teoremlar mavjudlik va yagonalik teoremlari deb yuritiladi. Keyinchalik, $f(x, y)$ funksiyaga ayrim shartlar qo'yish natijasida (0.0.5), (0.0.6) Koshi masalasining yechimi mavjud va yagonaligini ko'rsatamiz.

I-BOB. BIRINCHI TARTIBLI DIFFERENSIAL TENGLAMALAR

1-§. O'zgaruvchilari ajraladigan differensial tenglamalar

Dastavval, ayrim sodda differensial tenglamaning umumiy yechimini topish bilan shug'ullanamiz.

Ushbu

$$y' = f(x) \cdot g(y) \quad (1.1.1)$$

ko'rinishdagi differensial tenglamaga o'zgaruvchilari ajraladigan differensial tenglama deyiladi. Bu yerdagi $f(x)$ va $g(y)$ funksiyalar mos ravishda $a < x < b$ va $c < y < d$ oraliqlarda aniqlangan uzluksiz deb qaraladi. Bundan ko'rinadiki, (1.1.1) differensial tenglamaning o'ng tomoni quyidagi

$$D = (a, b) \times (c, d) = \{(x, y) \in R^2 : a < x < b, c < y < d\}$$

sohada aniqlangan va uzluksizdir. (1.1.1) ko'rinishdagi differensial tenglamaning yechimini topish uchun quyidagi ikki holni ko'rib chiqamiz:

1-hol. Aytaylik, $g(y) \neq 0$, $y \in (c, d)$ bo'lsin. U holda (1.1.1) differensial tenglamani ushbu

$$\frac{dy}{g(y)} = f(x)dx$$

ko'rinishda yozish mumkin. Bu tenglikning ikkala tomonini integrallab

$$\int \frac{dy}{g(y)} = \int f(x)dx \quad (1.1.2)$$

munosabatni hosil qilamiz. Ma'lumki, $[g(y)]^{-1}$ va $f(x)$ funksiyalar uzluksiz ekanligidan, ularning mos ravishda $G(y)$ va $F(x)$ boshlang'ich funksiyalarining mavjudligi kelib chiqadi. Shuning uchun (1.1.2) tenglikni quyidagi

$$G(y) = F(x) + C, \quad \forall C = const \quad (1.1.3)$$

ko'rinishda yozish mumkin. Qaralayotgan $g(y) \neq 0$ holda $G(y)$ monoton funksiya bo'ladi. Chunki,

$$G'(y) = \frac{1}{g(y)} \neq 0.$$

Bundan esa uning teskarisi G^{-1} mavjud ekanligi kelib chiqadi. Yuqoridagi (1.1.3) tenglikdan

$$y(x) = G^{-1}(F(x) + C) \quad (1.1.4)$$

funksiyani topamiz. O'z navbatida bu funksiya qaralayotgan holda (1) differensial tenglamaning umumiy yechimini ifodalaydi.

2-hol. Aytaylik biror $y(x) = \bar{y} \in (c, d)$ nuqtada $g(\bar{y}) = 0$ bo'lsin. Bu tenglamaning ildizi yordamida aniqlangan $y(x) = \bar{y}$ o'zgarmas funksiya (1.1.1) differensial tenglamaning yechimidan iborat bo'ladi.

Demak, (1.1.1) differensial tenglamaning umumiy yechimi

$$y(x) = \begin{cases} G^{-1}(F(x) + C), & \text{agar } g(y) \neq 0, \\ \bar{y}, & \text{agar } g(\bar{y}) = 0 \end{cases} \quad (1.1.5)$$

ko'rinishda bo'lar ekan.

Endi, tayinlangan biror $(x_0, y_0) \in D$ nuqtani olib, (1.1.1) differensial tenglamaning ushbu

$$y(x_0) = y_0 \quad (1.1.6)$$

boshlang'ich shartni qanoatlantiruvchi yechimini topish bilan shug'ullanamiz. Shu maqsadda quyidagi

$$F(x) = \int_{x_0}^x f(t) dt, \quad G(y) = \int_{y_0}^y \frac{1}{g(t)} dt \quad (1.1.7)$$

funksiyalarni tuzib olamiz.

Ushbu

$$f(x, y) = G(y) - F(x)$$

yordamchi funksiyani qaraylik. Ko'rinish turibdiki,

$$f(x_0, y_0) = G(y_0) - F(x_0) = 0$$

shart bajariladi. Aniqlanishiga ko'ra $G(y)$ va $F(x)$ uzluksiz hamda differensiullanuvchidir. Shuning uchun $f(x, y)$ ham D sohada uzluksiz va differensiullanuvchi bo'lib,

$$f'_y(x, y) = G'(y) = \frac{1}{g(y)},$$

$$f'_y(x_0, y_0) \neq 0$$

munosabatlarni qanoatlantiradi. Yuqoridagi mulohazalardan ko'rinadiki, $f(x, y)$ oshkormas funksiyani mavjudligi haqidagi teoremaning barcha shartlarini qanoatlantiradi:

1. $(x_0, y_0) \in D$ nuqtaning atrofida $f(x, y)$ differensiullanuvchi.
2. $f(x_0, y_0) = 0, f'_y(x_0, y_0) \neq 0$.

Bundan $f(x, y) = 0$ tenglama $x = x_0$ nuqtaning biror atrofida aniqlangan differensiullanuvchi va ushbu $y(x_0) = y_0$ shartni qanoatlantiruvchi $y = y(x)$ ildizining mavjudligi kelib chiqadi. Shu bilan bir qatorda

$$\frac{dy}{dx} = -\frac{f'_x(x, y)}{f'_y(x, y)} = \frac{F'(x)}{G'(y)} = f(x)g(y)$$

tenglikning o‘rinli bo‘lishi ham kelib chiqadi. Ko‘rinib turibdiki, $y(x)$ funksiya (1.1.1) differensial tenglamani va (1.1.6) boshlang‘ich shartni qanoatlantiruvchi xususiy yechimini ifodalaydi.

Mustaqil yechish uchun mashqlar [21], §2, №60-65

2-§. O‘zgaruvchilari ajraladigan differensial tenglamaga keltiriladigan differensial tenglamalar

Ushbu

$$y' = f(ax + by + c) \quad (1.2.1)$$

ko‘rinishdagi differensial tenglamada

$$z = ax + by + c \quad (1.2.2)$$

almashtirish bajarsak, o‘zgaruvchilari ajraladigan differensial tenglama hosil bo‘ladi. Bu yerda a, b, c – o‘zgarmas sonlar. Haqiqatan ham

$$z' = a + by' \quad (1.2.3)$$

bo‘lgani uchun (1.2.1) differensial tenglama quyidagi

$$z' = a + bf(z)$$

ko‘rinishni oladi. Bu esa o‘zgaruvchilari ajraladigan differensial tenglamadir.

3-§. Bir jinsli va kvazi bir jinsli differensial tenglamalar

1.3.1-ta’rif. Agar quyidagi

$$y' = f(x, y) \quad (1.3.1)$$

differensial tenglamaning o‘ng tomonidagi $f(x, y)$ funksiya uchun

$$f(x, y) = f(\lambda x, \lambda y), \quad \forall \lambda > 0 \quad (1.3.2)$$

shart bajarilsa, (1.3.1) differensial tenglamaga bir jinsli differensial tenglama deyiladi. Oxirgi (1.3.2) tenglikda $\lambda = \frac{1}{x}$ desak,

$$f(x, y) = f\left(1, \frac{y}{x}\right) := h\left(\frac{y}{x}\right)$$

munosabat hosil bo‘ladi. Buning natijasida (1.3.1) differensial tenglama ushbu

$$y' = h\left(\frac{y}{x}\right) \quad (1.3.3)$$

ko‘rinishni oladi. Endi (1.3.3) ko‘rinishdagi differensial tenglamaning yechimini topish bilan shug‘ullanamiz. Buning uchun quyidagi

$$y(x) = z(x) \cdot x \quad (1.3.4)$$

almashtirishdan foydalanamiz. Bu yerda $z = z(x)$ yangi noma'lum funksiya. Bu (1.3.4) almashtirishning ikkala tomonini differensiallab,

$$y' = z'x + z \quad (1.3.5)$$

tenglikni hosil qilamiz. (1.3.4) va (1.3.5) tengliklardan foydalanib, (1.3.3) differensial tenglamani quyidagicha yozish mumkin:

$$z'x + z = h(z),$$

ya'ni

$$z' = \frac{1}{x}[h(z) - z], \quad x \neq 0. \quad (1.3.6)$$

Bu esa o'zgaruvchilari ajraladigan differensial tenglamadir.

1-hol. Aytaylik, $h(z)$ funksiya $a < z < b$ intervalda uzluksiz bo'lib, $h(z) \neq z$ shartni qanoatlantirsin. U holda (1.3.6) differensial tenglamani o'zgaruvchilarni ajratish usulidan foydalanib yechish mumkin:

$$\frac{dz}{h(z) - z} = \frac{dx}{x},$$

$$x = C \cdot \exp \left\{ \int \frac{dz}{h(z) - z} \right\}.$$

Bu yerda $C \neq 0$ ixtiyoriy o'zgarmas son. Oxirgi tenglikda $z = \frac{y}{x}$ almashtirishga qaytib, (1.3.3) differensial tenglamaning umumiy yechimini hosil qilamiz.

2-hol. Aytaylik $z = z_0$ soni $h(z) - z = 0$ tenglamaning ildizi bo'lsin. Bu holda $y(x) = z_0 x$ funksiya (1.3.3) differensial tenglamaning yechimidan iborat bo'ladi.

1.3.2-ta'rif. Agar $f(x, y)$ funksiya uchun

$$f(\lambda x, \lambda y) = \lambda^k f(x, y), \quad \forall \lambda > 0 \quad (1.3.7)$$

shart bajarilsa, (1.3.1) tenglamaga k - darajali bir jinsli differensial tenglama deyiladi.

1.3.3-ta'rif. Agar $f(x, y)$ funksiya uchun

$$f(\lambda^\alpha x, \lambda^\beta y) = \lambda^{\beta - \alpha} f(x, y), \quad \forall \lambda > 0, \quad \alpha, \beta \in R \quad (1.3.8)$$

shart bajarilsa, (1.3.1) tenglamaga kvazi bir jinsli differensial tenglama deyiladi.

Oxirgi (1.3.8) holda ham (1.3.1) differensial tenglamani ushbu

$$y(x) = x^{\beta/\alpha} \cdot z(x) \quad (1.3.9)$$

almashtirish yordamida o'zgaruvchilari ajraladigan differensial tenglamaga keltirish mumkin. Buning uchun (1.3.8) tenglikda $\lambda = x^{-1/\alpha}$ deb

$$f(1, y/x^{\beta/\alpha}) = x^{1 - \beta/\alpha} f(x, y),$$

ya'ni

$$f(x, y) = x^{(\beta/\alpha)-1} f(1, y/x^{\beta/\alpha})$$

munosabatlarni topamiz. Oxirgi tenglikdan va (1.3.9) almashtirishdan foydalanib, (1.3.1) differensial tenglamani

$$x^{\beta/\alpha} \frac{dz}{dx} + \frac{\beta}{\alpha} z \cdot x^{-1+\beta/\alpha} = x^{-1+\beta/\alpha} f(1, z)$$

ko‘rinishga keltirish mumkin. Bundan

$$x \frac{dz}{dx} = f(1, z) - \frac{\beta}{\alpha} z \quad (1.3.10)$$

ko‘rinishdagi differensial tenglama kelib chiqadi. Bu esa o‘zgaruvchilari ajraladigan differensial tenglamadir.

1-hol. Aytaylik $f(1, z) - \frac{\beta}{\alpha} z \neq 0$ bo‘lsin. Bu holda (1.3.10) differensial tenglamadan

$$\frac{dx}{x} = \frac{dz}{f(1, z) - \frac{\beta}{\alpha} z}$$

munosabatni topamiz va uni integrallab, ushbu

$$x = C \cdot \exp \left\{ \int \frac{dz}{f(1, z) - \frac{\beta}{\alpha} z} \right\}$$

yechimni hosil qilamiz. Bu yerda $C \neq 0$ ixtiyoriy o‘zgarmas son. Oxirgi tenglikda

$$z = y \cdot x^{-\beta/\alpha}$$

almashtirishga qaytib

$$y' = x^{\beta/\alpha-1} f(1, y/x^{\beta/\alpha}) \quad (1.3.11)$$

ko‘rinishdagi differensial tenglamaning umumiy yechimini topamiz.

2-hol. Faraz qilaylik biror $z = z_0$ soni $f(1, z) - \frac{\beta}{\alpha} z = 0$ tenglamaning ildizi bo‘lsin. Bu holda ushbu

$$y = x^{\beta/\alpha} \cdot z_0$$

funksiya (1.3.11) differensial tenglamaning yechimi bo‘ladi.

Mustaqil yechish uchun mashqlar [21], §4, №101-112

4-§. Bir jinsli differensial tenglamaga keltiriladigan differensial tenglamalar

Ushbu

$$y' = f\left(\frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}\right) \quad (1.4.1)$$

koʻrinishdagi differensial tenglamaning umumiy yechimini topish uchun, uni oʻzgaruvchilari ajraladigan yoki bir jinsli differensial tenglamalarga keltiramiz. Buning uchun quyidagi hollarni koʻrib chiqamiz:

1-hol. Aytaylik $c_1 = c_2 = 0$ boʻlsin. Bu holda (1.4.1) differensial tenglama

$$y' = f\left(\frac{a_1x + b_1y}{a_2x + b_2y}\right) \quad (1.4.2)$$

koʻrinishni oladi. Oxirgi (1.4.2) differensial tenglamani ushbu

$$y' = f\left(\frac{a_1 + b_1 \frac{y}{x}}{a_2 + b_2 \frac{y}{x}}\right) \equiv h\left(\frac{y}{x}\right), \quad x \neq 0 \quad (1.4.3)$$

koʻrinishda yozish mumkin. Bu esa bir jinsli differensial tenglamadir.

2-hol. Aytaylik c_1, c_2 oʻzgarmas sonlarning kamida bittasi noldan farqli boʻlib, quyidagi

$$\begin{aligned} a_1x + b_1y + c_1 &= 0, \\ a_2x + b_2y + c_2 &= 0 \end{aligned} \quad (1.4.4)$$

toʻgʻri chiziqlar (x_0, y_0) nuqtada kesishsin. U holda koordinatalar boshini (x_0, y_0) nuqtaga koʻchirsak, berilgan differensial tenglama bir jinsli differensial tenglamaga keltiriladi. Haqiqatan ham, (1.4.1) differensial tenglamani ushbu

$$y' = f\left(\frac{a_1(x - x_0) + b_1(y - y_0)}{a_2(x - x_0) + b_2(y - y_0)}\right)$$

koʻrinishda yozib,

$$z = y - y_0, \quad t = x - x_0$$

almashtirish bajarsak, $y' = \dot{z}$ ekanligidan quyidagi

$$\dot{z} = f\left(\frac{a_1t + b_1z}{a_2t + b_2z}\right), \quad \dot{z} = \frac{dz}{dt}$$

differensial tenglama hosil boʻladi. Bu esa bir jinsli differensial tenglamadir.

3-hol. Faraz qilaylik, c_1, c_2 sonlarning kamida bittasi noldan farqli boʻlib, (1.4.4) toʻgʻri chiziqlar oʻzaro parallel boʻlsin. U holda

$$a_2 = ka_1, \quad b_2 = kb_1$$

munosabatlar bajarilgani uchun (1.4.1) differensial tenglama quyidagi ko‘rinishni oladi:

$$y' = f\left(\frac{a_1x + b_1y + c_1}{k(a_1x + b_1y) + c_2}\right).$$

Bu differensial tenglama $z = a_1x + b_1y$ almashtirish yordamida o‘zgaruvchilari ajraladigan differensial tenglamaga keltiriladi. Haqiqatan ham, quyidagi

$$\begin{aligned} z &= a_1x + b_1y, \\ z' &= a_1 + b_1y' \end{aligned}$$

belgilashlar natijasida

$$z' = a_1 + b_1f\left(\frac{z + c_1}{kz + c_2}\right)$$

differensial tenglamaga ega bo‘lamiz. Bu esa o‘zgaruvchilari ajraladigan differensial tenglamadir.

Mustaqil yechish uchun mashqlar [21], §4, №113-129.

5-§. Birinchi tartibli chiziqli differensial tenglama

Ushbu

$$y' = a(x)y + b(x) \quad (1.5.1)$$

ko‘rinishdagi tenglamaga birinchi tartibli chiziqli differensial tenglama deyiladi. Bu yerda $a(x)$ va $b(x)$ funksiyalar biror $[a, b] \subset R$ oraliqda aniqlangan va uzluksiz deb qaraladi.

Agar $b(x) \neq 0, x \in [a, b]$ bo‘lsa, (1.5.1) tenglamaga chiziqli bir jinsli bo‘lmagan differensial tenglama deyiladi. Agar $b(x) = 0, x \in [a, b]$ bo‘lsa, (1.5.1) tenglamaga chiziqli bir jinsli differensial tenglama deyiladi va ushbu

$$y' = a(x)y \quad (1.5.2)$$

ko‘rinishni oladi. Bu esa o‘zgaruvchilari ajraladigan differensial tenglamadir. Ko‘rinib turibdiki $y(x) = 0$ funksiya (1.5.2) differensial tenglamaning yechimidan iborat. Agar $y(x) \neq 0$ bo‘lsa, (1.5.2) differensial tenglamani quyidagicha yozish mumkin:

$$\frac{dy}{y} = a(x)dx.$$

Bu tenglikning ikkala tomonini integrallab, quyidagi

$$\ln|y| = \int_{x_0}^x a(t)dt + \ln|C_1| \quad (1.5.3)$$

tenglikni olamiz, bu yerda $C_1 \neq 0$ - ixtiyoriy haqiqiy son, $x_0 \in [a, b]$ - tayinlangan son. Ushbu

$$A(x) = \int_{x_0}^x a(t) dt$$

belgilashdan foydanib, (1.5.3) tenglikdan

$$y(x) = C_1 \exp \left\{ \int_{x_0}^x a(t) dt \right\} = C_1 e^{A(x)} \quad (1.5.4)$$

formulani hosil qilamiz. Bu yerda $x_0, x \in [a, b]$, C_1 -ixtiyoriy o'zgarmas son desak, (1.5.4) formula (1.5.2) ko'rinishdagi bir jinsli differensial tenglamaning umumiy yechimini ifodalaydi.

Bir jinsli bo'lmagan (1.5.1) ko'rinishdagi differensial tenglamaning umumiy yechimini topishning bir qancha usullari bor. Avvalo biz Lagranj, ya'ni o'zgarmasni variatsiyalash usuli bilan tanishamiz. Shu maqsadda (1.5.1) differensial tenglamaning yechimini ushbu

$$y(x) = C(x)e^{A(x)} \quad (1.5.5)$$

ko'rinishda izlaymiz. Bu yerda $C(x)$ – hozircha noma'lum funksiya. (1.5.5) tenglikning ikki tomonini differensiallab

$$y' = C'(x)e^{A(x)} + C(x)e^{A(x)}a(x) \quad (1.5.6)$$

tenglikni hosil qilamiz. Bu y va y' funksiyalarning (1.5.5) va (1.5.6) ifodalarini mos ravishda (1.5.1) differensial tenglamaga qo'yib

$$C'(x)e^{A(x)} + a(x)C(x)e^{A(x)} = a(x)C(x)e^{A(x)} + b(x)$$

munosabatni topamiz. Bundan

$$C'(x)e^{A(x)} = b(x)$$

kelib chiqadi. Oxirgi tenglikni

$$C'(x) = b(x)e^{-A(x)}$$

ko'rinishda yozib, uni integrallasak

$$C(x) = \int_{x_0}^x b(t)e^{-A(t)} dt + C_1, \quad C_1 = const \quad (1.5.7)$$

munosabatni hosil qilamiz. Yuqoridagi (1.5.5) tenglikdan va (1.5.7) formuladan foydalanib, (1.5.1) differensial tenglamaning umumiy yechimini topamiz:

$$y(x) = \left\{ C_1 + \int_{x_0}^x b(t) \cdot e^{-A(t)} dt \right\} e^{A(x)} = C_1 e^{A(x)} + e^{A(x)} \cdot \int_{x_0}^x b(t) \cdot e^{-A(t)} dt. \quad (1.5.8)$$

Bu formuladan foydalanib (1.5.1) differensial tenglamaning

$$y(x_0) = y_0 \quad (1.5.9)$$

boshlang'ich shartni qanoatlantiruvchi yechimini ham topish mumkin:

$$y(x) = y_0 e^{A(x)} + e^{A(x)} \int_{x_0}^x b(t) \cdot e^{-A(t)} dt. \quad (1.5.10)$$

Bu yerda x_0 va y_0 berilgan sonlar. Agar (1.5.8) tenglikning o'ng tomonidagi ikkinchi hadni

$$\tilde{y}(x) = e^{A(x)} \int_{x_0}^x b(t) \cdot e^{-A(t)} dt \quad (1.5.11)$$

belgilab olsak, u holda $\tilde{y}(x)$ funksiya (1.5.1) differensial tenglamaning

$$\tilde{y}(x_0) = 0 \quad (1.5.12)$$

boshlang'ich shartni qanoatlantiruvchi xususiy yechimini beradi. Shuning uchun (1.5.8) formula

$$y(x) = C_1 e^{A(x)} + \tilde{y}(x) \quad (1.5.13)$$

ko'rinishni oladi. Bu esa bir jinsli bo'lmagan (1.5.1) differensial tenglamaning umumiy yechimi bir jinsli (1.5.2) differensial tenglamaning $C_1 e^{A(x)}$ umumiy yechimi bilan bir jinsli bo'lmagan (1.5.1) differensial tenglamaning $\tilde{y}(x)$ xususiy yechimining yig'indisidan iborat ekanligini ko'rsatadi.

Endi, (1.5.1) ko'rinishdagi chiziqli differensial tenglamaning umumiy yechimini topishning Bernulli usuli bilan tanishamiz. Shu maqsadda (1.5.1) differensial tenglamaning yechimini

$$y(x) = u(x) \cdot v(x) \quad (1.5.14)$$

ko'rinishda izlaymiz. Natijada biz ushbu

$$v \frac{du}{dx} + u \frac{dv}{dx} = a(x)uv + b(x),$$

ya'ni

$$\left(\frac{dv}{dx} - a(x)v \right) u + v \frac{du}{dx} = b(x) \quad (1.5.15)$$

ko'rinishidagi differensial tenglamaga ega bo'lamiz. Bunda $v(x)$ funksiyani shunday tanlaymizki, natijada

$$\frac{dv}{dx} - a(x)v = 0$$

shart bajarilsin. Bu esa o'zgaruvchilari ajraladigan differensial tenglamadir. Bu tenglamani yechib,

$$v(x) = e^{\int_{x_0}^x a(t) dt} = e^{A(x)} \quad (1.5.16)$$

funksiyani topamiz. Shuning uchun (1.5.15) differensial tenglama ushbu

$$\frac{du}{dx} = b(x) e^{-A(x)}$$

ko'rinishni oladi. Bu differensial tenglamani integrallab,

$$u(x) = C_0 + \int_{x_0}^x b(t) \cdot e^{-A(t)} dt, C_0 = const \quad (1.5.17)$$

munosabatni hosil qilamiz. Topilgan $u(x)$ va $v(x)$ funksiyaning (1.5.16) va (1.5.17) ifodalarni (1.5.14) tenglikka qo'yib,

$$y(x) = C_0 e^{A(x)} + e^{A(x)} \int_{x_0}^x b(t) e^{-A(t)} dt$$

(1.5.1) differensial tenglamaning umumiy yechimini olamiz.

Endi, bir jinsli bo'lmagan (1.5.1) differensial tenglamaning xususiy yechimini topishning Koshi usuli bilan tanishamiz. Shu maqsadda, biror $x = t \in [a, b]$ nuqtani olib, quyidagi bir jinsli differensial tenglamaga qo'yilgan

$$\begin{cases} y' = a(x)y, \\ y(x)|_{x=t} = 1 \end{cases} \quad (1.5.18)$$

Koshi masalasining yechimini topamiz:

$$y(x, t) = \exp \left\{ \int_t^x a(\tau) d\tau \right\}. \quad (1.5.19)$$

Bundan foydalanib ushbu

$$\tilde{y}(x) = \int_{x_0}^x y(x, t) b(t) dt \quad (1.5.20)$$

funksiyani tuzib olamiz. Ko'rinib turibdiki bu funksiya

$$y(x_0) = 0 \quad (1.5.21)$$

boshlang'ich shartni qanoatlantiradi. Yuqoridagi (1.5.20) tenglikning ikkala tomonini differensiallab

$$\begin{aligned} \tilde{y}'(x) &= y(x, x)b(x) + \int_{x_0}^x y'_x(x, t)b(t)dt = \\ &= 1 \cdot b(x) + a(x) \int_{x_0}^x \exp \left\{ \int_t^x a(\tau) d\tau \right\} b(t) dt = \\ &= b(x) + a(x) \int_{x_0}^x y(x, t) b(t) dt = b(x) + a(x) \tilde{y}(x) \end{aligned}$$

(1.5.1) ko'rinishdagi differensial tenglamani keltirib chiqaramiz. Bundan ko'rinadiki (1.5.20) tenglik orqali aniqlangan $\tilde{y}(x)$ funksiya (1.5.1) differensial tenglamaning xususiy yechimini berar ekan. Bundan foydalanib (1.5.13) tenglikdan

$$y(x) = C_1 e^{A(x)} + \int_{x_0}^x \exp\left\{\int_t^x a(\tau) d\tau\right\} b(t) dt$$

(1.5.1) differensial tenglamaning umumiy yechimining yana bir (Koshi) ko'rishini topamiz.

1.5.1-izoh. Agar (1.5.1) differensial tenglamaning bitta xususiy yechimi ma'lum bo'lsa, u holda uning umumiy yechimi bitta kvadratura yordamida topiladi.

Isbot. Aytaylik $y = y_1(x)$ funksiya (1.5.1) differensial tenglamaning xususiy yechimi bo'lsin. U holda (1.5.1) tenglama

$$y_1' = a(x)y_1 + b(x)$$

ayniyatga aylanadi. Bu tenglikni (1.5.1) tenglamadan ayirsak, quyidagi

$$\frac{d}{dx}(y - y_1) = a(x)(y - y_1)$$

munosabat hosil bo'ladi. Bundan

$$y(x) - y_1(x) = C \exp\left\{\int a(x) dx\right\}$$

kelib chiqadi. Bu yerda C -ixtiyoriy o'zgarmas son. ■

1.5.2-izoh. Agar (1.5.1) differensial tenglamaning ikkita xususiy yechimi ma'lum bo'lsa, u holda uning umumiy yechimi kvadraturasiz topiladi.

Isbot. Faraz qilaylik $y = y_1(x)$ va $y = y_2(x)$ funksiyalar (1.5.1) differensial tenglamaning xususiy yechimlari bo'lsin. U holda bu xususiy yechimlarni ushbu

$$y_1(x) = C_1 f(x) + g(x), \quad y_2(x) = C_2 f(x) + g(x)$$

ko'rinishda yozish mumkin. (1.5.1) differensial tenglamaning umumiy yechimini

$$y(x) = C f(x) + g(x)$$

ko'rinishda yozilar edi. Bu yerda C -ixtiyoriy o'zgarmas son.

Yuqoridagi tengliklardan

$$\frac{y(x) - y_1(x)}{y_2(x) - y_1(x)} = \frac{C - C_1}{C_2 - C_1}$$

munosabat kelib chiqadi. Bundan quyidagi

$$y(x) = \frac{C - C_1}{C_2 - C_1} (y_2(x) - y_1(x)) + y_1(x)$$

tenglikni olamiz. ■

1.5.3-Izoh. Agar $y_1(x), y_2(x), y_3(x)$ funksiyalar (1.5.1) differensial tenglamaning yechimlari bo'lsa, u holda

$$\frac{y_3(x) - y_1(x)}{y_3(x) - y_2(x)} = \text{const}$$

munosabat o‘rinli bo‘ladi.

Haqiqatan ham, (1.5.13) ga asosan $y_j(x)$, $j = \overline{1,3}$ yechimlarni ushbu

$$y_j(x) = c_j f(x) + g(x), \quad j = \overline{1,3}$$

ko‘rinishda yozish mumkin. Bundan foydalanib quyidagi ifodaning qiymatini topamiz:

$$\frac{y_3 - y_1}{y_3 - y_2} = \frac{c_3 f + g - c_1 f - g}{c_3 f + g - c_2 f - g} = \frac{(c_3 - c_1)f}{(c_3 - c_2)f} = \text{const.} \quad \blacksquare$$

Mustaqil yechish uchun mashqlar [21], §5, №136-150.

6-§. Noma‘lum koeffitsiyentlar usuli

Ushbu

$$y' - \lambda y = P_m(x)e^{\mu x} \quad (1.6.1)$$

ko‘rinishdagi differensial tenglamaning xususiy yechimini topishning aniqmas koeffitsiyentlar usuli bilan tanishamiz. Bu yerda $\lambda, \mu = \text{const}$, $P_m(x)$ – darajasi m ga teng bo‘lgan ko‘phad.

1.6.1-teorema. 1) Agar $\mu \neq \lambda$ bo‘lsa, u holda (1.6.1) differensial tenglamaning xususiy yechimi

$$y(x) = Q_m(x)e^{\mu x} \quad (1.6.2)$$

ko‘rinishda bo‘ladi.

2) Agar $\mu = \lambda$ bo‘lsa, u holda (1.6.1) differensial tenglamaning xususiy yechimi

$$y(x) = xQ_m(x)e^{\mu x} \quad (1.6.3)$$

ko‘rinishda bo‘ladi. Bunda $Q_m(x)$ – darajasi m ga teng bo‘lgan ko‘phad.

Isbot. Berilgan (1.6.1) differensial tenglamaning yechimini

$$y(x) = e^{\mu x} z, \quad z = z(x) \quad (1.6.4)$$

ko‘rinishda izlaymiz. Bu tenglikning ikki tomonini differensiallab,

$$y' = \mu e^{\mu x} z + e^{\mu x} z' \quad (1.6.5)$$

munosabatni topamiz. (1.6.4) va (1.6.5) tengliklarga asosan (1.6.1) differensial tenglamani quyidagicha yozish mumkin:

$$e^{\mu x} z' + \mu e^{\mu x} z - \lambda e^{\mu x} z = P_m(x)e^{\mu x}.$$

Oxirgi tenglikning ikki tomonini $e^{\mu x}$ ga bo‘lib,

$$z' + (\mu - \lambda)z = P_m(x) \quad (1.6.6)$$

differensial tenglamani hosil qilamiz. Bu tenglamada $\mu = \lambda$ bo‘lsa, (1.6.6) differensial tenglama

$$z' = P_m(x) \quad (1.6.7)$$

ko‘rinishni oladi. Tenglamaning o‘ng tomonidagi ko‘phad

$$P_m(x) = a_0 x^m + a_1 x^{m-1} + \dots + a_m$$

ko‘rinishda bo‘lgani uchun (1.6.7) differensial tenglamaning yechimi

$$z(x) = \int_0^x P_m(t) dt = \frac{a_0}{m+1} x^{m+1} + \frac{a_1}{m} x^m + \dots + a_m x =$$

$$= x \left(\frac{a_0}{m+1} x^m + \frac{a_1}{m} x^{m-1} + \dots + a_m \right) = x Q_m(x)$$

ko‘rinishni oladi. Bundan va yuqoridagi (1.6.4) almashtirishga asosan (1.6.1) differensial tenglamaning yechimi $\mu = \lambda$ holda

$$y(x) = x Q_m(x) e^{\mu x}$$

ko‘rinishda bo‘lishi kelib chiqadi.

Agar (1.6.6) differensial tenglamada $\mu \neq \lambda$ bo‘lsa, u holda uning yechimini

$$z(x) = b_0 x^m + b_1 x^{m-1} + \dots + b_{m-1} x + b_m \quad (1.6.8)$$

ko‘rinishda izlaymiz. Bu yerda $b_j, j = 0, 1, 2, \dots, m -$ hozircha noma‘lum sonlar. (1.6.8) tenglikni differensiallab,

$$z' = m b_0 x^{m-1} + b_1 (m-1) x^{m-2} + \dots + b_{m-1} \quad (1.6.9)$$

munosabatni hosil qilamiz. (1.6.8) va (1.6.9) tengliklardan foydalanib, (1.6.6) differensial tenglamani quyidagicha yozish mumkin:

$$m b_0 x^{m-1} + b_1 (m-1) x^{m-2} + \dots + b_{m-1} + (\mu - \lambda)(b_0 x^m + b_1 x^{m-1} + \dots + b_m) =$$

$$= a_0 x^m + a_1 x^{m-1} + \dots + a_m,$$

ya‘ni

$$(\mu - \lambda) b_0 x^m + [(\mu - \lambda) b_1 + b_0 m] x^{m-1} + \dots + b_{m-1} + (\mu - \lambda) b_m =$$

$$= a_0 x^m + a_1 x^{m-1} + \dots + a_m. \quad (1.6.10)$$

Bu yerda ko‘phadlarning tengligidan foydalansak, $b_j, j = 0, 1, 2, \dots, m$ noma‘lumlariga nisbatan quyidagi

$$(\mu - \lambda) b_0 = a_0,$$

$$(\mu - \lambda) b_1 + b_0 m = a_1,$$

$$\dots \dots \dots$$

$$b_{m-1} + (\mu - \lambda) b_m = a_m$$

tenglamalar sistemasi hosil bo‘ladi. Bu tenglamalarni ketma-ket yechib,

$$b_0 = \frac{a_0}{\mu - \lambda}, b_1 = \frac{a_1 - m b_0}{\mu - \lambda}, \dots \quad (1.6.11)$$

noma‘lumlarini aniqlaymiz. Bundan ko‘rinadiki, b_0, b_1, \dots, b_m koeffitsiyentlar ketma-ket yagona aniqlanadi. Shunday qilib, (1.6.11) munosabatlarni inobatga olsak, (1.6.8) tenglik quyidagi ko‘rinishni oladi:

$$z(x) = \frac{a_0}{\mu - \lambda} x^m + \frac{a_1 - m b_0}{\mu - \lambda} x^{m-1} + \dots$$

Endi, (1.6.4) almashtirishdan foydalanib, (1.6.1) differensial tenglamaning $\mu \neq \lambda$ holdagi xususiy yechimini olamiz:

$$y(x) = e^{\mu x} \left[\frac{a_0}{\mu - \lambda} x^m + \frac{a_1 - mb_0}{\mu - \lambda} x^{m-1} + \dots \right]. \blacksquare$$

7-§. Bernulli differensial tenglamasi

Ushbu

$$y' = a(x)y + b(x)y^n \quad (1.7.1)$$

ko'rinishdagi tenglamaga Bernulli differensial tenglamasi deyiladi. Bu yerda $a(x), b(x) \in C(\alpha, \beta)$, ya'ni (α, β) intervalda aniqlangan uzluksiz funksiyalar.

Agar $n = 0$ bo'lsa, u holda

$$y' = a(x)y + b(x)$$

chiziqli differensial tenglama hosil bo'ladi.

Agar $n = 1$ bo'lsa, u holda

$$y' = [a(x) + b(x)]y$$

bir jinsli chiziqli differensial tenglama hosil bo'ladi.

Aytaylik, $n \neq 0, n \neq 1$ bo'lsin. Ko'rinib turibdiki, $y = 0$ (1.7.1)

differensial tenglamaning yechimidan iborat. Agar $y \neq 0$ bo'lsa, u holda (1.7.1) tenglamaning ikki tomonini y^n ga bo'lib ushbu

$$y^{-n} y' = a(x)y^{1-n} + b(x) \quad (1.7.2)$$

differensial tenglamani hosil qilamiz. Bunda

$$z = y^{1-n} \quad (1.7.3)$$

almashtirishni bajaramiz. Quyidagi

$$z' = (1-n)y^{-n}y', \quad y^{-n}y' = \frac{1}{1-n}z'$$

munosabatlardan foydalanib (1.7.2) tenglamani ushbu

$$\frac{1}{1-n}z' = a(x)z + b(x),$$

ya'ni

$$z' = (1-n)a(x)z + (1-n)b(x) \quad (1.7.4)$$

ko'rinishda yozish mumkin. Bu esa chiziqli bir jinsli bo'lmagan differensial tenglamadir.

Mustaqil yechish uchun mashqlar [21], §5, №150-160.

8-§. Rikkati differensial tenglamasi

Ushbu

$$y' = a(x)y^2 + b(x)y + c(x) \quad (1.8.1)$$

ko`rinishdagi tenglamaga Rikkati differensial tenglamasi deyiladi. Bu yerda $a(x), b(x), c(x) \in C(\alpha, \beta), (-\infty \leq \alpha < \beta \leq \infty)$ bo`lib, $a(x) \neq 0, c(x) \neq 0$.

Agar $a(x) = 0$ bo`lsa, u holda (1.8.1) differensial tenglama ushbu

$$y' = b(x)y + c(x)$$

ko`rinishni oladi. Bu esa chiziqli bir jinsli bo`lmagan differensial tenglamadir.

Agar $c(x) = 0$ bo`lsa, u holda (1.8.1) differensial tenglama

$$y' = a(x)y^2 + b(x)y$$

ko`rinishni oladi. Bu esa Bernulli differensial tenglamasidir.

Umumiy holda Rikkati differensial tenglamasi kvadraturada integrallanmaydi.

Shuni alohida qayd qilish lozimki, ayrim xususiy hollardagina Rikkati differensial tenglamasini kvadraturada integrallanishini ko`rsatish mumkin.

Jumladan 1841 yilda Liuvill ushbu

$$y' = Ay^2 + Bx^\alpha, \quad A \neq 0, B \neq 0$$

ko`rinishdagi Rikkati differensial tenglamasi kvadraturada integrallanuvchi bo`lishi uchun $\frac{\alpha}{(2\alpha + 4)}$ soni butun bo`lishi kerakligini ko`rsatib berdi.

Endi Rikkati differensial tenglamasining ayrim xossalarini o`rganishga o`tamiz.

1.8.1-lemma. Rikkati tenglamasi quydagi:

1. $x = \varphi(\bar{x}),$

2. Kasr-chiziqli

$$y = \frac{\alpha(x)y_1 + \beta(x)}{\gamma(x)y_1 + \delta(x)}, \quad \alpha(x)\delta(x) - \beta(x)\gamma(x) \neq 0$$

amashtirishlarga nisbatan ko`rinishini o`zgartirmaydi.

Isbot. 1. Ushbu $x = \varphi(\bar{x})$ tenglikning ikki tomonini differensiallab,

$$dx = \varphi'(\bar{x})d\bar{x}, \quad y' = \frac{dy}{\varphi'(\bar{x})d\bar{x}}$$

munosabatlarni topamiz. Bu tengliklarni (1.8.1) ga qo`yib,

$$\frac{dy}{d\bar{x}} = a(\varphi(\bar{x}))\varphi'(\bar{x})y^2 + b(\varphi(\bar{x}))\varphi'(\bar{x})y + c(\varphi(\bar{x}))\varphi'(\bar{x}) \quad (1.8.2)$$

differensial tenglamani hosil qilamiz. Bunda ushbu

$$A(\bar{x}) = a(\bar{x})\varphi'(\bar{x}), \quad B(\bar{x}) = b(\varphi(\bar{x}))\varphi'(\bar{x}), \quad C(\bar{x}) = c(\varphi(\bar{x}))\varphi'(\bar{x})$$

belgilashlardan foydalansak, (1.8.2) tenglama

$$\frac{dy}{dx} = A(\bar{x})y^2 + B(\bar{x})y + C(\bar{x})$$

ko`rinishni oladi. Bu esa Rikkati differensial tenglamasidir.

2. Berilgan kasr-chiziqli almashtirishning ikki tomonini differensiallab,

$$y' = \frac{(\alpha\delta - \beta\gamma)y_1' + (\alpha'\gamma - \alpha\gamma')y_1^2 + (\alpha'\delta + \beta'\gamma - \alpha\delta' - \gamma'\beta)y_1 + \beta'\delta - \beta\delta'}{(\gamma y_1 + \delta)^2} \quad (1.8.3)$$

differensial tenglamani topamiz. Berilgan kasr-chiziqli almashtirish natijasida ushbu

$$a(x)y^2 + b(x)y + c(x)$$

kvadrat uchhadning o'zgarishini aniqlaymiz:

$$\begin{aligned} & a(x)y^2 + b(x)y + c(x) = \\ & = \frac{a(x)(\alpha y_1 + \beta)^2 + b(x)(\alpha y_1 + \beta)(\gamma y_1 + \delta) + c(x)(\gamma y_1 + \delta)^2}{(\gamma y_1 + \delta)^2}. \end{aligned} \quad (1.8.4)$$

Yuqoridagi (1.8.1) differensial tenglamadan va (1.8.3) hamda (1.8.4) munosabatlardan foydalanib quyidagi

$$\begin{aligned} & (\alpha\delta - \beta\gamma)y_1' + (\alpha'\gamma - \alpha\gamma')y_1^2 + (\alpha'\delta + \beta'\gamma - \alpha\delta' - \beta\gamma')y_1 + \beta'\delta - \delta'\beta = \\ & = a(x)(\alpha y_1 + \beta)^2 + b(x)(\alpha y_1 + \beta)(\gamma y_1 + \delta) + c(x)(\gamma y_1 + \delta)^2 \end{aligned}$$

tenglikni hosil qilamiz. Bu tenglik elementar amallar natijasida ushbu

$$\begin{aligned} & (\alpha\delta - \beta\gamma)y_1' = (a\alpha^2 + b\alpha\gamma + c\gamma^2 - \alpha'\gamma + \alpha\gamma')y_1^2 + [2\alpha\beta a + (\alpha\delta + \beta\gamma)b + 2\gamma\delta c - \\ & - \alpha'\delta - \beta'\gamma + \alpha\delta' + \beta\gamma'] \cdot y_1 + a\beta^2 + b\beta\delta + c\delta^2 - \beta'\delta + \delta'\beta \end{aligned}$$

ko'rinishni oladi. Bundan

$$\begin{aligned} y_1' & = \frac{a\alpha^2 + b\alpha\gamma + c\gamma^2 - \alpha'\gamma + \alpha\gamma'}{\alpha\delta - \beta\gamma} y_1^2 + \\ & + \frac{[2\alpha\beta a + (\alpha\delta + \beta\gamma)b + 2\gamma\delta c - \alpha'\delta - \beta'\gamma + \alpha\delta' + \beta\gamma']}{\alpha\delta - \beta\gamma} y_1 + \\ & + \frac{a\beta^2 + b\beta\delta + c\delta^2 - \beta'\delta + \delta'\beta}{\alpha\delta - \beta\gamma} \end{aligned}$$

kelib chiqadi. Bu esa Rikkati differensial tenglamasidir. ■

1.8.2-lemma. Rikkati tenglamasini ushbu

$$y = \varphi(x)z, \quad z = \omega + \psi(x), \quad x = -\tilde{x} \quad (1.8.5)$$

almashtirishlar yordamida

$$y' = y^2 + f(x). \quad (1.8.6)$$

ko'rinishga keltirish mumkin.

Isbot. Avvalo (1.8.5) almashtirishdan

$$y' = \phi'z + \phi z' \quad (1.8.7)$$

tenglikni topamiz. So'ngra (1.8.5) va (1.8.7) larni (1.8.1) differensial tenglamaga qo'yib,

$$\phi'z + \phi z' = a\phi^2 z^2 + b\phi z + c,$$

ya'ni

$$z' = a\phi z^2 + \left(b - \frac{\phi'}{\phi} \right) z + \frac{c}{\phi} \quad (1.8.8)$$

differensial tenglamani olamiz. Bu yerda

$$\varphi = \frac{1}{a}$$

deb tanlansa, u holda

$$\varphi' = -\frac{a'}{a^2}$$

o'rinli bo'ladi. Natijada (1.8.8) differensial tenglama

$$z' = z^2 + \left(b + \frac{a'}{a}\right)z + ca \quad (1.8.9)$$

ko'rinishni oladi. Bu differensial tenglamada ushbu

$$z = \omega + \psi$$

almashtirishni bajarib

$$\psi' = \omega^2 + 2\omega\psi + \psi^2 + \left(b + \frac{a'}{a}\right)\omega + \left(b + \frac{a'}{a}\right)\psi + ca - \omega',$$

ya'ni

$$\psi' = \psi^2 + \left(b + \frac{a'}{a} + 2\omega\right)\psi + \omega^2 + \left(b + \frac{a'}{a}\right)\omega + ca - \omega' \quad (1.8.10)$$

differensial tenglamani topamiz. Oxirgi (1.8.10) tenglamada ψ oldidagi koeffitsiyentni nolga tenglashtirsak,

$$b + \frac{a'}{a} + 2\omega = 0,$$

ya'ni

$$\omega = -\frac{ab + a'}{2a}$$

kelib chiqadi. Natijada (1.8.10) differensial tenglama

$$\psi' = \psi^2 + f, \quad f = \omega^2 + \left(b + \frac{a'}{a}\right)\omega + ca - \omega'$$

kanonik ko'rinishga keladi. Lemma isbotlandi. ■

1.8.1-teorema. Agar Rikkati tenglamasining bitta xususiy yechimi ma'lum bo'lsa, u holda Rikkati tenglamasining barcha yechimlari ikkita kvadratura yordamida topiladi.

Isbot. Faraz qilaylik, $y_1 = y_1(x)$ funksiya (1.8.1) differensial tenglamaning xususiy yechimi bo'lsin. U holda

$$y = y_1 + z$$

almashtirish natijasida (1.8.1) tenglama ushbu

$$\begin{aligned} y_1' + z' &= a(x)(y_1 + z)^2 + b(x)(y_1 + z) + c(x) = \\ &= a(x)y_1^2 + b(x)y_1 + c(x) + a(x)z^2 + 2a(x)y_1z + b(x)z \end{aligned} \quad (1.8.11)$$

ko'rinishni oladi. Teorema shartiga ko'ra

$$y_1' = a(x)y_1^2 + b(x)y_1 + c(x)$$

o‘rinli. Bundan foydalanib (1.8.11) tenglamani

$$z' = a(x)z^2 + (b(x) + 2a(x)y_1)z \quad (1.8.12)$$

ko‘rinishda yozish mumkin. Bu esa Bernulli differensial tenglamasidir, uning yechimi ikkita kvadratura yordamida topiladi. Chunki (1.8.12) tenglama

$$z = \frac{1}{u}$$

almashtirish yordamida chiziqli differensial tenglamaga keladi. Shunday qilib

$$y = \frac{1}{u} + y_1, \quad u = \frac{1}{y - y_1} \quad (1.8.13)$$

almashtirish natijasidan Rikkati differensial tenglamasi chiziqli differensial tenglamaga keltirilgan ekan. Teorema isbotlandi. ■

1.8.2-teorema. Agar Rikkati tenglamasining ikkita xususiy yechimi ma'lum bo'lsa, u holda uning umumiy yechimi bitta kvadratura yordamida topiladi.

Isbot. Aytaylik $y_1 = y_1(x)$ va $y_2 = y_2(x)$ funksiyalar (1.8.1) differensial tenglamaning xususiy yechimlari bo'lsin. U holda (1.8.12) differensial tenglamani

$$z = \frac{1}{u}$$

almashtirish yordamida

$$u' + (b + 2ay_1)u + a = 0 \quad (1.8.14)$$

chiziqli differensial tenglamaga keltiramiz. (1.8.13) munosabatga asosan (1.8.14) tenglamaning bitta xususiy yechimi

$$u = \frac{1}{y_2 - y_1}$$

bo'ladi. Bu holda (1.8.14) tenglamaning yechimi bitta kvadratura yordamida topiladi. Teorema isbot bo'ldi. ■

1.8.3-teorema. Agar Rikkati tenglamasining uchta xususiy yechimi ma'lum bo'lsa, u holda uning umumiy yechimi kvadraturasiz topiladi.

Isbot. Faraz qilaylik, $y_1 = y_1(x)$, $y_2 = y_2(x)$ va $y_3 = y_3(x)$ funksiyalar (1.8.1) differensial tenglamaning xususiy yechimlari bo'lsin. U holda (1.8.14) chiziqli differensial tenglama ikkita

$$u_1 = \frac{1}{y_2 - y_1}, \quad u_2 = \frac{1}{y_3 - y_1}$$

xususiy yechimlarga ega bo'ladi. Shuning uchun (1.8.14) chiziqli differensial tenglamaning umumiy yechimi kvadraturasiz topiladi:

$$u = \frac{1}{y_2 - y_1} + c \left(\frac{1}{y_2 - y_1} - \frac{1}{y_3 - y_1} \right). \quad (1.8.15)$$

(1.8.13) va (1.8.15) tengliklarni tenglashtirib

$$\frac{1}{y-y_1} = \frac{1}{y_2-y_1} + c \left(\frac{1}{y_2-y_1} - \frac{1}{y_3-y_1} \right)$$

munosabatni hosil qilamiz. Bundan o'z navbatida o'zgarmas c sonining qiymati topiladi:

$$c = \frac{y-y_2}{y-y_1} : \frac{y_3-y_2}{y_3-y_1}. \quad (1.8.16)$$

Bu esa Rikkati tenglamasining umumiy integralidir. Teorema isbotlandi. ■

1.8.1-natija. Agar Rikkati tenglamasining to'rtta $y_1 = y_1(x)$, $y_2 = y_2(x)$, $y_3 = y_3(x)$, $y_4 = y_4(x)$ xususiy yechimlari ma'lum bo'lsa, u holda quyidagi

$$\frac{y_4-y_2}{y_4-y_1} : \frac{y_3-y_2}{y_3-y_1} = c = const$$

ayniyat o'rinli bo'ladi.

1.8.4-teorema. Rikkati tenglamasining umumiy yechimi, ixtiyoriy o'zgarmas C sonining kasr-chiziqli almashtirishidan iborat.

Isbot. Yuqoridagi (1.8.14) chiziqli differensial tenglamaning umumiy yechimi

$$u = Cf(x) + g(x)$$

ko'rinishga ega bo'lganligidan, (1.8.13) almashtirishni

$$y = y_1 + \frac{1}{Cf(x) + g(x)} = \frac{y_1f(x)C + y_1g(x) + 1}{Cf(x) + g(x)},$$

ya'ni

$$y(x) = \frac{C\varphi_1(x) + \varphi_2(x)}{C\psi_1(x) + \psi_2(x)} \quad (1.8.17)$$

ko'rinishda yozish mumkin. Bu yerda

$$\varphi_1(x) = f(x)y_1, \quad \varphi_2(x) = y_1g(x) + 1, \quad \psi_1(x) = f(x), \quad \psi_2(x) = g(x).$$

Bundan ko'rinadiki, (1.8.17) tenglik yordamida aniqlangan $y(x)$ funksiya C ning kasr-chiziqli almashtirishidan iborat. ■

Mustaqil yechish uchun mashqlar [21], §5, №167-171.

9-§. Rikkati tenglamasining maxsus ko'rinishi

Ushbu

$$y' = Ay^2 + Bx^m \quad (1.9.1)$$

tenglamaga Rikkati tenglamasining maxsus ko'rinishi deyiladi. Bu yerda A, B va m o'zgarmas sonlar. Biz m sonining qanday qiymatlarida (1.9.1) differensial tenglamaning umumiy yechimi elementar funksiyalarda topilishi mumkinligini o'rganamiz.

Avvalo eng sodda hollarni qaraymiz:

1. Aytaylik, $m=0$ bo'lsin. Bu holda (1.9.1) differensial tenglama

$$y' = Ay^2 + B \quad (1.9.2)$$

ko'rinishga keladi. Bundan ushbu

$$\frac{dy}{Ay^2 + B} = dx, \quad Ay^2 + B \neq 0$$

tenglikni topamiz. Bu esa (1.9.2) differensial tenglamaning umumiy yechimi elementar funksiyalarda ifodalanishini ko'rsatadi.

2. Aytaylik, $m = -2$ bo'lsin. Bu holda (1.9.1) differensial tenglama

$$y' = Ay^2 + \frac{B}{x^2} \quad (1.9.3)$$

ko'rinishni oladi. Ushbu

$$y = \frac{1}{z}$$

almashtirishdan foydalansak, (1.9.3) differensial tenglama

$$-\frac{dz}{dx} = B\left(\frac{z}{x}\right)^2 + A \quad (1.9.4)$$

ko'rinishni oladi. Bu esa bir jinsli differensial tenglamadir. Quyidagi

$$u = \frac{z}{x}$$

almashtirish natijasida (1.9.4) differensial tenglama o'zgaruvchilari ajraladigan

$$-xu' = Bu^2 + u + A$$

differensial tenglamaga keladi. Bundan ko'rinadiki, (1.9.3) differensial tenglamaning umumiy yechimi elementar funksiyalar orqali ifodalanadi.

3. Agar

$$\frac{m}{2m+4} \in Z \setminus \{0; -2\}$$

butun son bo'lsa, (1.9.1) differensial tenglama kvadraturada integrallanadi.

10-§. Rikkati va ikkinchi tartibli chiziqli differensial tenglama orasidagi bog'lanishlar

1. Avvalo, ushbu

$$y' = a(x)y^2 + b(x)y + c(x) \quad (1.10.1)$$

Rikkati differensial tenglamasida quyidagi

$$y = \frac{z(x)}{a(x)}, \quad a(x) \neq 0 \quad (1.10.2)$$

almashtirish bajaramiz. Buning uchun (1.10.2) tenglikning ikki tarafini differensiallab,

$$y' = -\frac{a'}{a^2}z + \frac{1}{a}z' \quad (1.10.3)$$

topamiz. (1.10.2) va (1.10.3) tengliklardan foydalanib, (1.10.1) differensial tenglamani quyidagicha yozamiz:

$$-\frac{a'}{a^2}z + \frac{1}{a}z' = a\frac{z^2}{a^2} + b\frac{z}{a} + c.$$

Bu tenglamaning ikki tomonini a^2 ga ko'paytirib,

$$az' - a'z = az^2 + abz + a^2c$$

munosabatni olamiz. Bundan

$$z' = z^2 + \left(b + \frac{a'}{a}\right)z + ac$$

kelib chiqadi. Bu differensial tenglamani

$$z' - z^2 + P(x)z + Q = 0, \quad (1.10.4)$$

$$P(x) = -\left(b + \frac{a'}{a}\right), \quad Q(x) = -ac$$

ko'rinishda yozib olamiz va

$$z = -\frac{u'}{u} \quad (1.10.5)$$

almashtirish bajaramiz. Bu tenglikning ikki tomonini differensiallab,

$$z' = -\frac{u''u - u'^2}{u^2} = -\frac{u''}{u} + \left(\frac{u'}{u}\right)^2$$

topamiz. Oxirgi tenglikdan va (1.10.5) almashtirishdan foydalanib, (1.10.4) tenglamani

$$-\frac{u''}{u} + \left(\frac{u'}{u}\right)^2 - \left(\frac{u'}{u}\right)^2 - P\frac{u'}{u} + Q = 0,$$

$$-\frac{u''}{u} - P\frac{u'}{u} + Q = 0,$$

ya'ni

$$u'' + Pu' - Qu = 0 \quad (1.10.6)$$

ko'rinishda yozish mumkin. Bu esa ikkinchi tartibli chiziqli differensial tenglamadir.

2. Ko'p hollarda, ayniqsa tatbiqiy ahamiyatga ega bo'lgan masalalarda ikkinchi tartibli chiziqli differensial tenglamaning ayrim yechimlarini topish va uning xossalarini o'rganish uchun uni Rikkati tenglamasiga keltiriladi. Shu maqsadda ushbu

$$y'' - q(x)y = 0 \quad (1.10.7)$$

chiziqli differensial tenglamada

$$\omega = \frac{y'}{y} \quad (1.10.8)$$

almashtirish bajaramiz. Buning uchun (1.10.8) tenglikning ikki tomonini differensiallab,

$$\omega' = \frac{y''}{y} - \left(\frac{y'}{y}\right)^2 = \frac{y''}{y} - \omega^2 \quad (1.10.9)$$

munosabatni hosil qilamiz. (1.10.7) tenglamadan

$$\frac{y''}{y} = q(x)$$

ekanligini hisobga olsak, (1.10.9) differensial tenglama

$$\omega' + \omega^2 = q(x) \quad (1.10.10)$$

ko'rinishni oladi. Bu esa Rikkati differensial tenglamasidir.

11-§. To'liq differensialli tenglamalar

Hosilaga nisbatan yechilgan birinchi tartibli differensial tenglamaning ushbu

$$M(x, y)dx + N(x, y)dy = 0 \quad (1.11.1)$$

ko'rinishi bilan tanishamiz. Bu yerda $M(x, y)$ va $N(x, y)$ funksiyalar $\Gamma \subseteq R^2$ sohada aniqlangan uzluksiz, ya'ni

$$M(x, y), N(x, y) \in C(\Gamma).$$

1.11.1-ta'rif. Agar (1.11.1) differensial tenglamaning chap tomoni biror $U(x, y) \in C^{(1)}(\Gamma)$ funksiyaning to'liq differensialidan iborat bo'lsa, u holda (1.11.1) tenglamaga to'liq differensialli tenglama deyiladi.

Agar (1.11.1) tenglama to'liq differensialli tenglama bo'lsa, u holda uning chap tomoni

$$M(x, y)dx + N(x, y)dy = dU(x, y) \quad (1.11.2)$$

ko'rinishda yoziladi. Bu holda $y = \varphi(x) \in C^{(1)}(a, b)$ funksiya (1.11.1) differensial tenglamaning yechimi bo'lishi uchun ushbu

$$U(x, y)|_{y=\varphi(x)} = U(x, \varphi(x)) = C = const, \quad \forall x \in (a, b) \quad (1.11.3)$$

shartning bajarilishi zarur va yetarlidir. Chunki (1.11.2) tenglikda

$$dU = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy \quad (1.11.4)$$

munosabatdan foydalansak, undan

$$\frac{\partial U}{\partial x} = M(x, y), \quad \frac{\partial U}{\partial y} = N(x, y) \quad (1.11.5)$$

tengliklarni olamiz.

Faraz qilaylik, $y = \varphi(x) \in C^{(1)}(a, b)$ funksiya (1.11.1) to'liq differensialli tenglamaning yechimi bo'lsin. U holda

$$M(x, \varphi(x))dx + N(x, \varphi(x))\varphi'(x)dx = 0,$$

y'ani

$$M(x, \varphi(x)) + N(x, \varphi(x))\varphi'(x) = 0$$

kelib chiqadi. Bu yerda (1.11.2) dan foydalanib,

$$\frac{d}{dx}U(x, \varphi(x)) = 0$$

tenglikni topamiz. Bundan

$$U(x, \varphi(x)) = C = \text{const}$$

kelib chiqadi.

Agar $y = \varphi(x)$ funksiya

$$U(x, \varphi(x)) = C$$

tenglamaning yechimi bo'lsa, u holda uni differensiallab,

$$M(x, \varphi(x)) + N(x, \varphi(x))\varphi'(x) = 0$$

tenglikni hosil qilamiz. Bundan esa $y = \varphi(x)$ funksiya (1.11.1) differensial tenglamaning yechimi ekanligi kelib chiqadi.

1.11.1-teorema. Agar bir bog'lamli Γ sohada $M(x, y), N(x, y)$, $\frac{\partial M}{\partial y}, \frac{\partial N}{\partial x}$ funksiyalar aniqlangan va uzluksiz bo'lib,

$$M^2(x, y) + N^2(x, y) \neq 0, \forall x, y \in \Gamma \quad (1.11.6)$$

shart bajarilsa, u holda (1.11.1) to'liq differensialli tenglama bo'lishi uchun, ushbu

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad (1.11.7)$$

shartning bajarilishi zarur va yetarli.

Isbot. Zaruriyligi. Faraz qilaylik, (1.11.1) tenglama to'liq differensialli tenglama bo'lsin. U holda (1.11.5) munosabatlar bajariladi. Bunda ushbu

$$\frac{\partial^2 U}{\partial x \partial y} = \frac{\partial^2 U}{\partial y \partial x}$$

aralash hosilalarning tengligini inobatga olsak,

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

kelib chiqadi.

Yetarliligi. Aytaylik, $\Gamma \subseteq R^2$ sohaning har bir nuqtasida (1.11.7) shart bajarilsin. U holda (1.11.1) ning to'liq differensialli tenglama ekanligini isbotlaymiz. Buning uchun (1.11.5) tengliklarni qanoatlantiruvchi $U(x, y)$ funksiyani topamiz. Ushbu

$$\frac{\partial U}{\partial x} = M(x, y)$$

tenglikni integrallab,

$$U(x, y) = \int_{x_0}^x M(x, y)dx + \varphi(y) \quad (1.11.8)$$

munosabatni hosil qilamiz. Bu tenglikning ikki tomonini y o'zgaruvchi bo'yicha differensiallab, so'ngra (1.11.7) munosabatdan foydalansak

$$\begin{aligned} \frac{\partial U}{\partial y} &= \int_{x_0}^x M'_y(x, y)dx + \varphi'(y) = \int_{x_0}^x N'_x(x, y)dx + \varphi'(y) = \\ &= N(x, y) - N(x_0, y) + \varphi'(y) \end{aligned} \quad (1.11.9)$$

tenglikni olamiz. Bunda φ funksiyani shunday tanlaymizki, natijada quyidagi

$$\varphi'(y) = N(x_0, y) \quad (1.11.10)$$

tenglik bajarilsin. U holda (1.11.9) tenglik

$$\frac{\partial U}{\partial y} = N(x, y)$$

ko'rinishni oladi. Endi (1.11.10) tenglikni qanoatlantiruvchi birorta $\varphi(y)$ funksiyani topamiz:

$$\varphi(y) = \int_{y_0}^y N(x_0, t)dt, \quad \varphi(y_0) = 0. \quad (1.11.11)$$

Bu formuladan foydalanib, (1.11.8) tenglikni quyidagicha yozish mumkin:

$$U(x, y) = \int_{x_0}^x M(x, y)dx + \int_{y_0}^y N(x_0, t)dt.$$

Teorema isbot bo'ldi. ■

Mustaqil yechish uchun mashqlar [21], §6, №186-194.

12-§. Integrallovchi ko'paytuvchi

Faraz qilaylik, ushbu

$$M(x, y)dx + N(x, y)dy = 0 \quad (1.12.1)$$

tenglama to'liq differensialli tenglama bo'lmasin, ya'ni $\Gamma \subseteq R^2$ sohada aniqlangan birorta ham $U(x, y)$ funksiya uchun

$$dU = M(x, y)dx + N(x, y)dy$$

tenglik o'rinli bo'lmasin.

1.12.1-ta'rif. Agar $\Gamma \subseteq R^2$ sohada berilgan $M(x, y), N(x, y)$ va birorta $\mu(x, y) \neq 0$ funksiya uchun, ushbu

$$\mu(x, y)M(x, y)dx + \mu(x, y)N(x, y)dy = 0 \quad (1.12.2)$$

tenglama to'liq differensialli tenglama bo'lsa, u holda (1.12.1) tenglamaga to'liq differensialli tenglamaga keltiriladigan tenglama, $\mu(x, y)$ funksiyaga esa uning integrallovchi ko'paytuvchisi deyiladi. Bu holda

$$\mu M dx + \mu N dy = dU \quad (1.12.3)$$

o'rinli bo'ladi. Bundan

$$\mu M = \frac{\partial U}{\partial x}, \quad \mu N = \frac{\partial U}{\partial y} \quad (1.12.4)$$

ekanligini topamiz.

1.12.1-teorema. Agar $0 \neq \mu(x, y) \in C^{(1)}(\Gamma)$, $M(x, y) \in C^{(1)}(\Gamma)$, $N(x, y) \in C^{(1)}(\Gamma)$ bo'lib, $y = y(x)$, $y(x_0) = y_0$ funksiya $x_0 \in I = (a, b)$ intervalda aniqlangan hamda (2) differensial tenglamaning yechimi bo'lsa, u holda $y = y(x)$ funksiya (1.12.1) differensial tenglamaning ham shu $I = (a, b)$ intervalda aniqlangan yechimi bo'ladi.

Isbot. Teorema shartiga ko'ra, $y = y(x)$ funksiya (1.12.2) differensial tenglamaning yechimi bo'lgani uchun, ushbu

$$\mu(x, y(x))M(x, y(x)) + \mu(x, y(x))N(x, y(x))y'(x) = 0, \quad x_0 \in I = (a, b) \quad (1.12.5)$$

tenglik o'rinli bo'ladi. Bu tenglikda $\mu(x, y(x)) \neq 0$ ekanligini inobatga olsak, (1.12.5) tenglama

$$M(x, y(x)) + N(x, y(x))y'(x) = 0, \quad x \in I = (a, b)$$

ko'rinishni oladi. Bu esa o'z navbatida $y = y(x)$ funksiya (1.12.1) differensial tenglamaning yechimi ekanligini ko'rsatadi. ■

Endi integrallovchi ko'paytuvchining ayrim xossalari bilan tanishamiz. Aytaylik, (1.12.2) tenglama to'liq differensialli tenglama bo'lsin. Boshqacha aytganda $\mu(x, y) \neq 0$ funksiya (1.12.1) tenglamaning integrallovchi ko'paytuvchisi bo'lsin. U holda (1.12.4) tengliklardan

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x} \quad (1.12.6)$$

munosabatni topamiz. Bu tenglikni quyidagi ko'rinishda yozamiz:

$$M \frac{\partial \mu}{\partial y} + \mu \frac{\partial M}{\partial y} = N \frac{\partial \mu}{\partial x} + \mu \frac{\partial N}{\partial x}$$

yoki

$$M \frac{\partial \mu}{\partial y} - N \frac{\partial \mu}{\partial x} = \mu \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right). \quad (1.12.7)$$

Bunda $\mu(x, y) > 0$, $\forall (x, y) \in \Gamma \subseteq R^2$ deb olsak, (1.12.7) dan

$$M \frac{\partial \ln \mu}{\partial y} - N \frac{\partial \ln \mu}{\partial x} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \quad (1.12.8)$$

kelib chiqadi. Bu munosabat $\ln \mu(x, y)$ funksiyaga nisbatan birinchi tartibli bir jinsli bo'lmagan xususiy hosilali differensial tenglamadir. Bizga (1.12.8)

tenglamaning biror xususiy yechimini topish yetarlidir. Bunday yechim $(x_0, y_0) \in \Gamma$ nuqtaning yetarli kichik atrofida M, N, N'_x, M'_y funksiyalar uzluksiz bo'lgani uchun mavjud.

1.12.2-teorema. Agar (1.12.1) differensial tenglama $U(x, y) = C$ umumiy integralga ega bo'lsa, u holda (1.12.1) differensial tenglama uchun integrallovchi ko'paytuvchi mavjud bo'ladi.

Isbot. Teorema shartiga ko'ra $U(x, y) = C$, (1.12.1) differensial tenglamaning umumiy integrali bo'lgani uchun

$$dU = 0, \quad (1.12.9)$$

ya'ni

$$\frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy = 0 \quad (1.12.10)$$

tenglik o'rinli bo'ladi. Bu yerda

$$\frac{\partial U}{\partial y} \neq 0, \quad (x, y) \in \Gamma$$

desak, (1.12.10) tenglamadan

$$\frac{dy}{dx} = -\frac{\frac{\partial U}{\partial x}}{\frac{\partial U}{\partial y}} \quad (1.12.11)$$

kelib chiqadi. Ikkinchi tomondan, (1.12.1) differensial tenglamaga asosan

$$\frac{dy}{dx} = -\frac{M}{N} \quad (1.12.12)$$

munosabatni hosil qilamiz. (1.12.11) va (1.12.12) tengliklarni o'zaro tenglashtirib

$$\frac{M}{N} = \frac{\frac{\partial U}{\partial x}}{\frac{\partial U}{\partial y}},$$

ya'ni

$$\frac{\frac{\partial U}{\partial x}}{M} = \frac{\frac{\partial U}{\partial y}}{N} = \mu$$

bo'lishini topamiz. Bundan

$$\frac{\partial U}{\partial x} = \mu M, \quad \frac{\partial U}{\partial y} = \mu N$$

kelib chiqadi. O'z navbatida ushbu

$$\mu[Mdx + Ndy] = \mu Mdx + \mu Ndy = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy = dU$$

munosabatlardan $\mu(x, y)$ funksiya (1.12.1) differensial tenglama uchun integrallovchi ko'paytuvchi ekanligi kelib chiqadi.

Endi integrallovchi ko'paytuvchini topish bilan shug'ullanamiz. Yuqoridagi mulohazalardan ko'rinadiki, $\mu(x, y)$ funksiyani topish uchun (1.12.8) xususiy hosilali differensial tenglamani xususiy yechimini topish kerak bo'ladi. Bu masala o'z navbatida qo'yilgan masalaga nisbatan ham ancha murakkab masaladir. Ayrim hollarda $\mu(x, y)$ integrallovchi ko'paytuvchini topish uchun (1.12.7) yoki (1.12.8) tenglamalardan foydalansa bo'ladi.

1.12.3-teorema. Agar shunday $\exists \omega(x, y) \in C^{(1)}(\Gamma)$ funksiya mavjud bo'lib,

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N \frac{\partial \omega}{\partial x} - M \frac{\partial \omega}{\partial y}} = \psi(\omega), \quad \forall (x, y) \in \Gamma \quad (1.12.13)$$

tenglik o'rinli bo'lsa, u holda integrallovchi ko'paytuvchi

$$\mu(x, y) = \exp \left\{ \int \psi(\omega) d\omega \right\} \quad (1.12.14)$$

formula orqali topiladi.

Isbot. Yuqoridagi (1.12.7) differensial tenglamaning yechimini $\mu = \mu(\omega)$ ko'rinishda izlaymiz. U holda

$$d\mu = \frac{\partial \mu}{\partial x} dx + \frac{\partial \mu}{\partial y} dy = \frac{\partial \mu}{\partial \omega} \frac{\partial \omega}{\partial x} dx + \frac{\partial \mu}{\partial \omega} \frac{\partial \omega}{\partial y} dy = \frac{\partial \mu}{\partial \omega} \left(\frac{\partial \omega}{\partial x} dx + \frac{\partial \omega}{\partial y} dy \right) = \frac{\partial \mu}{\partial \omega} d\omega$$

hosil bo'ladi. Bundan

$$\frac{d\mu}{d\omega} = \frac{\partial \mu}{\partial \omega} \quad (1.12.15)$$

ekanligini topamiz. Bu holda (1.12.7) tenglama quydagi ko'rinishni oladi:

$$M \frac{\partial \mu}{\partial \omega} \frac{\partial \omega}{\partial y} - N \frac{\partial \mu}{\partial \omega} \frac{\partial \omega}{\partial x} = \mu \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right).$$

Bu yerda (1.12.15) tenglikni e'tiborga olsak, oxirgi tenglama ushbu

$$\frac{d\mu}{d\omega} \left(N \frac{\partial \omega}{\partial x} - M \frac{\partial \omega}{\partial y} \right) = \mu \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right)$$

ko'rinishga keladi. Bundan

$$\frac{d\mu}{\mu} = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N \frac{\partial \omega}{\partial x} - M \frac{\partial \omega}{\partial y}} d\omega$$

munosabat kelib chiqadi. (1.12.13) tenglikdan foydalanib, oxirgi munosabatni ushbu

$$\frac{d\mu}{\mu} = \psi(\omega)d\omega$$

ko‘rinishda yozish mumkin. Bu tenglamani integrallab

$$\mu = C \exp \left\{ \int \psi(\omega)d\omega \right\}$$

integrallovchi ko‘paytuvchini topamiz. Bizga birorta integrallovchi ko‘paytuvchi kerak. Shuning uchun $C = 1$ deb tanlash biz uchun yetarlidir:

$$\mu = \exp \left\{ \int \psi(\omega)d\omega \right\}.$$

Teorema isbotlandi. ■

1.12.1-misol. Chiziqli bir jinsli bo‘lmagan

$$\frac{dy}{dx} = p(x)y + q(x) \quad (1.12.16)$$

differensial tenglamaning integrallovchi ko‘paytuvchisini topamiz.

Yechish. Avvalo (1.12.16) differensial tenglamani

$$(p(x)y + q(x))dx - dy = 0$$

ko‘rinishda yozib olamiz. Bu holda

$$M(x, y) = p(x)y + q(x), \quad N(x, y) = -1$$

bo‘lgani uchun

$$\frac{\partial M}{\partial y} = p(x), \quad \frac{\partial N}{\partial x} = 0$$

munosabatlar o‘rinli. Ko‘rinib turibdiki,

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}.$$

Shuning uchun (1.12.16) chiziqli tenglama to‘liq differensialli tenglama emas.

Endi (1.12.16) differensial tenglamaning integrallovchi ko‘paytuvchisini

$$\mu = \mu(x)$$

ko‘rinishda izlaymiz. Bu holda $\omega = x$ bo‘lgani uchun

$$\psi(x) = \frac{M'_y - N'_x}{N} = -p(x)$$

tenglik o‘rinli bo‘ladi. (1.12.14) tenglikdan esa

$$\mu = \exp \left\{ - \int p(x)dx \right\}$$

formula kelib chiqadi.

1.12.4-teorema. Agar μ_0 (1.12.1) differensial tenglamaning integrallovchi ko‘paytuvchisi bo‘lib, $u_0(x, y)$ uning integrali bo‘lsa, u holda

$$\mu = \mu_0 \phi(u_0) \quad (1.12.17)$$

funksiya ham (1.12.1) tenglamaning integrallovchi ko‘paytuvchisi bo‘ladi. Bu yerda $\varphi \neq 0$ ixtiyoriy differensiallanuvchi funksiya.

Isbot. Berilgan (1.12.1) differensial tenglamaning chap tomonini $\mu_0\varphi(u_0)$ ga ko‘paytirib

$$\mu_0\varphi(u_0)(Mdx + Ndy) = \varphi(u_0)\mu_0(Mdx + Ndy) = \varphi(u_0)du_0 = d\left(\int \varphi(u_0)du_0\right)$$

munosabatni hosil qilamiz. Bundan (1.12.17) tenglik bilan aniqlangan μ funksiya (1.12.1) differensial tenglamaning integrallovchi ko‘paytuvchisi ekanligi kelib chiqadi. ■

Mustaqil yechish uchun mashqlar [21], §6, №195-220.

13-§. Koshi masalasi yechimining mavjudligi va yagonaligi

Hosilaga nisbatan yechilgan ushbu

$$y' = f(x, y) \quad (1.13.1)$$

differensial tenglamaning

$$y(x_0) = y_0 \quad (1.13.2)$$

boshlang‘ich shartni qanoatlantiruvchi $y = y(x)$ yechimini topishga Koshi masalasi deyiladi.

1.13.1-teorema (Koshi). Agar $f(x, y)$ funksiya

$$P = \{(x, y) \in R^2 : |x - x_0| \leq a, |y - y_0| \leq b\}$$

to‘g‘ri to‘rtburchakda aniqlangan va uzluksiz bo‘lib, y -o‘zgaruvchi bo‘yicha Lipshits shartini, ya‘ni $\forall (x, y_j) \in P, j=1,2$ nuqtalar uchun shunday $N > 0$ soni topilib

$$|f(x, y_1) - f(x, y_2)| \leq N|y_1 - y_2| \quad (1.13.3)$$

tengsizlikni qanoatlantirsa, u holda shunday $h > 0$ soni mavjudki, (1.13.1)-(1.13.2) Koshi masalasining $[x_0 - h, x_0 + h]$ oralig‘ida aniqlangan va (1.13.2) boshlang‘ich shartni qanoatlantiruvchi yagona $y = \varphi(x)$ yechimi mavjud bo‘ladi. Bu yerda

$$h = \min\left(a, \frac{b}{M}\right), \quad M = \max_{(x, y) \in P} |f(x, y)|. \quad (1.13.4)$$

1.13.1-izoh. Agar $f(x, y)$ funksiya P sohaning har bir nuqtasida $f'_y(x, y)$ xususiy hosilaga ega bo‘lib,

$$|f'_y(x, y)| \leq C, \quad C = const$$

shartni qanoatlantirsa, u holda bu funksiya P to‘g‘ri to‘rtburchakda y -o‘zgaruvchi bo‘yicha Lipshits shartini qanoatlantiradi.

Haqiqatan ham, ixtiyoriy ikki $(x, y_1), (x, y_2) \in P$ nuqtalar uchun Lagranj teoremasiga asosan quyidagi munosabat bajariladi:

$$|f(x, y_1) - f(x, y_2)| = |f'_y(x, y_1 + \theta(y_2 - y_1))| \cdot |y_2 - y_1|,$$

bu yerda $0 < \theta < 1$.

Oxirgi munosabatdan va $f'_y(x, y)$ xususiy hosilaning chegaralanganligidan (1.13.3) tengsizlik kelib chiqadi.

Ammo, ba'zi hollarda hosilaga ega bo'lmagan funksiyalar ham (1.13.3) Lipshits shartini qanoatlantiradi.

Masalan. Ushbu $f(x, y) = |y|$ funksiya $y = 0$, ya'ni $((x, 0))$ nuqtada hosilaga ega emas, lekin

$$|f(x, y_1) - f(x, y_2)| = ||y_1| - |y_2|| \leq |y_1 - y_2|$$

o'rinli. Bunda Lipshits o'zgarishi $N = 1$ bo'ladi.

Teoremani isbotlashdan oldin quyidagi misollarni qaraylik.

1.13.1-misol. Ushbu

$$y' = 3y^{\frac{2}{3}}, \quad y(1) = 0$$

Koshi masalasining yechimini toping.

Yechish. Berilgan differensial tenglamada o'zgaruvchilarni ajratib quyidagi

$$\frac{1}{3} y^{-\frac{2}{3}} dy = dx, \quad \frac{1}{3} \int y^{-\frac{2}{3}} dy = \int dx, \quad y^{\frac{1}{3}} = x + C, \quad y(x) = (x + C)^3, \quad C = const$$

yechimni topamiz. Boshlang'ich shartdan foydalanib,

$$y(1) = 0, \quad (1 + C)^3 = 0, \quad C = -1$$

berilgan Koshi masalasining

$$y(x) = (x - 1)^3$$

yechimini topamiz. Bundan tashqari, qaralayotgan Koshi masalasi $y(x) = 0$ yechimga ham ega. Demak, berilgan Koshi masalasi ikkita

$$y(x) = (x - 1)^3, \quad y(x) = 0$$

yechimga ega ekan. Bundan ko'rinadiki, berilgan differensial tenglamaning o'ng tomonidagi

$$f(x, y) = 3y^{\frac{2}{3}}$$

funksiya (1.13.3) - Lipshits shartini qanoatlantirmaydi. Chunki

$$f'_y|_{y=0} = \frac{2}{3\sqrt[3]{y}}|_{y=0} = +\infty.$$

Shuning uchun ham berilgan Koshi masalasining yechimi yagona emas.

1.13.2-misol. Ushbu Koshi masalasining yechimini toping:

$$y' = \sqrt[3]{y}, \quad y(0) = 0.$$

Yechish. Berilgan differensial tenglamada o'zgaruvchilarni ajratib, uning umumiy yechimini topamiz:

$$y^{-1/3} dy = dx, \int y^{-1/3} dy = \int dx, y^{2/3}(x) = \frac{2}{3}(x + C), C = \text{const.}$$

Endi boshlang'ich shartdan foydalanib C – o'zgarishning qiymatini aniqlaymiz:

$$y(0) = 0, 0 = \frac{2}{3}(0 + C), C = 0,$$

$$y^{2/3}(x) = \frac{2}{3}x, y(x) = \left(\frac{2x}{3}\right)^{3/2}, x > 0.$$

Ushbu

$$y(x) = \left(\frac{2x}{3}\right)^{3/2}, x > 0$$

funksiya berilgan Koshi masalasining yechimidan iborat bo'lar ekan. Bundan tashqari $y(x) = 0$ funksiya ham berilgan Koshi masalasining yechimi bo'ladi. Demak, berilgan Koshi masalasi ikkita yechimga ega ekan. Chunki, $f(x, y) = \sqrt[3]{y}$ funksiya $(x, 0)$ nuqtaning atrofida Lipshtits shartini qanoatlantirmaydi. Shuning uchun yechimning yagonaligi buziladi.

Quyidagi misolga e'tibor qarataylik.

1.13.3-misol. Ushbu Koshi masalasining yechimini toping:

$$y' = \frac{1}{y^2}, y(x_0) = 0.$$

Yechish. Bu misolda ham, berilgan differensial tenglamada o'zgaruvchilarni ajratib, uning umumiy yechimini topamiz:

$$y^2 dy = dx, \int y^2 dy = \int dx, y^3(x) = 3(x + C), y(x) = \sqrt[3]{3(x + C)}.$$

Boshlang'ich shartdan foydalanib C – o'zgarishning qiymatini aniqlaymiz:

$$y(x_0) = 0, x_0 + C = 0, C = -x_0.$$

Bundan ko'rinadiki, ushbu

$$y(x) = \sqrt[3]{3(x - x_0)}$$

funksiya berilgan Koshi masalasining yagona yechimidan iborat bo'ladi.

Shuni alohida qayd qilish lozimki, berilgan Koshi masalasidagi

$$f(x, y) = \frac{1}{y^2}, f'_y(x, y) = -\frac{2}{y^3}$$

funksiyalarning $(x_0, 0)$ nuqtada uzluksizligi buziladi. Ammo, berilgan Koshi masalasi yagona yechimga ega. Demak, Koshi teoremasidagi shartlar Koshi masalasi yechimi mavjud va yagona bo'lishi uchun yetarli shartlardir. Koshi masalasi yechimining yagonaligidan $f(x, y)$ funksiyaning uzluksizligi va y o'zgaruvchi bo'yicha Lipshtits shartini qanoatlantirishi kelib chiqmaydi.

Teoremaning isboti (Yechimning mavjudligi). Berilgan differensial tenglamani ushbu

$$dy = f(x, y)dx$$

ko‘rinishda yozib, uni (x_0, x) interval bo‘yicha integrallaymiz:

$$\int_{x_0}^x dy = \int_{x_0}^x f(t, y(t))dt.$$

Hosil bo‘lgan bu tenglikda (1.13.2) boshlang‘ich shartdan foydalanib,

$$y(x) = y_0 + \int_{x_0}^x f(t, y(t))dt \quad (1.13.5)$$

munosabatni hosil qilamiz. Bu munosabat $y(x)$ funksiyaga nisbatan integral tenglamadir. Shunday qilib, agar $y(x)$ funksiya (1.13.1)-(1.13.2) Koshi masalasining yechimi bo‘lsa, u holda $y(x)$ (1.13.5) integral tenglamani qanoatlantirar ekan. Aksincha, agar $y(x)$ uzluksiz funksiya (1.13.5) integral tenglamaning yechimi bo‘lsa, u holda $y(x)$ berilgan (1.13.1)-(1.13.2) Koshi masalasining ham yechimi bo‘lishini ko‘rsatish mumkin. Haqiqatan ham, $y(x)$ uzluksiz funksiya (1.13.5) integral tenglamani qanoatlantirsin. U holda $f(x, y(x))$ funksiya P – sohada uzluksiz bo‘lgani uchun

$$\frac{d}{dx} \left(\int_{x_0}^x f(t, y(t))dt \right) = f(x, y(x))$$

munosabatning o‘rinli bo‘lishi “Matematik analiz” fanidan ma’lum. Yuqoridagi (1.13.5) tenglikning ikki tomonini differensiallab

$$y'(x) = \left(y_0 + \int_{x_0}^x f(t, y(t))dt \right)' = f(x, y(x))$$

ekanligini topamiz. (1.13.2) boshlang‘ich shartning bajarilishi (1.13.5) tenglikdan ko‘rinib turibdi:

$$y(x_0) = y_0 + \int_{x_0}^{x_0} f(t, y(t))dt = y_0 + 0 = y_0.$$

Shunday qilib, (1.1.1)-(1.13.2) Koshi masalasi (1.13.5) integral tenglamaga ekvivalent ekan. Shuning ushun (1.13.1)-(1.13.2) Koshi masalasi yechimini mavjudligini ko‘rsatish o‘rniga, unga ekvivalent bo‘lgan (1.13.5) integral tenglama yechimini mavjudligini ko‘rsatamiz. Buning uchun ketma-ket yaqinlashishlar (Pikar) usulidan foydalanamiz.

Quyidagi

$$\begin{aligned}
y_0(x) &= y_0, \\
y_1(x) &= y_0 + \int_{x_0}^x f(t, y_0(t)) dt, \\
y_2(x) &= y_0 + \int_{x_0}^x f(t, y_1(t)) dt, \\
&\dots\dots\dots \\
y_n(x) &= y_0 + \int_{x_0}^x f(t, y_{n-1}(t)) dt, \\
&\dots\dots\dots
\end{aligned}
\tag{1.13.6}$$

formulalar yordamida $\{y_n(x)\}_{n=0}^{\infty}$ funksional ketma-ketlikni tuzib olamiz. Bu yerdagi $y_j(x)$, $j=0,1,2,\dots$ funksiyalarning har biri (1.13.2) boshlang'ich shartni, ya'ni $y_j(x_0) = y_0$, $j=0,1,2,\dots$ qanoatlantiradi.

Endi, ushbu

$$|y_1(x) - y_0|, |y_2(x) - y_0|, \dots, |y_n(x) - y_0|, \dots$$

ayirmalarni baholaymiz:

$$|y_1(x) - y_0| = \left| \int_{x_0}^x f(t, y_0(t)) dt \right| \leq \left| \int_{x_0}^x |f(t, y_0(t))| dt \right| \leq M \left| \int_{x_0}^x dt \right| = M |x - x_0|,$$

(1.13.7)

$$|y_n(x) - y_0| = \left| \int_{x_0}^x f(t, y_{n-1}(t)) dt \right| \leq \left| \int_{x_0}^x |f(t, y_{n-1}(t))| dt \right| \leq M \left| \int_{x_0}^x dt \right| = M |x - x_0|.$$

Bundan ko'rinadiki, agar x lar ushbu

$$|x - x_0| \leq h, \quad h = \min\left(a, \frac{b}{M}\right), \quad M = \max_{(x,y) \in P} |f(x, y)|,$$

tengsizlikni qanoatlantirsa, u holda (1.13.7) bahodan

$$|y_1(x) - y_0| \leq M \cdot h \leq M \cdot \frac{b}{M} = b, \tag{1.13.7'}$$

$$|y_n(x) - y_0| \leq M \cdot h \leq M \cdot \frac{b}{M} = b, \quad n = 1, 2, \dots$$

tengsizliklar kelib chiqadi. Bu esa $y = y_n(x)$, $n=0,1,2,\dots$ funksiyalarning grafiklari $\forall x \in \{x: |x - x_0| \leq h\}$ larda P to'g'ri to'rtburchakdan chiqib ketmasligini ko'rsatadi. Shunday qilib, $x_0 - h \leq x \leq x_0 + h$ tengsizlik bajarilsa, $y_n(x)$, $n=0,1,2,\dots$ funksiyalarning grafiklari

$$(x, y_n(x)) \in P, \quad n = 1, 2, \dots$$

P to'g'ri to'rtburchakda joylashar ekan.

Endi, har bir tayinlangan $x \in [x_0 - h, x_0 + h]$ larda ushbu $\{y_n(x)\}_{n=0}^{\infty}$ sonli ketma-ketlikning $n \rightarrow \infty$ da chekli limiti mavjudligini ko'rsatamiz va uni

$$\lim_{n \rightarrow \infty} y_n(x) = y(x) \in R \quad (1.13.8)$$

orqali belgilaymiz. Shu maqsadda, matematik induksiya usulini qo'llab

$$|y_n(x) - y_{n-1}(x)| \leq MN^{n-1} \frac{|x - x_0|^n}{n!} \quad (1.13.9)$$

bahoning o'rinli bo'lishini ko'rsatamiz. Bu baho $n=1$ da o'rinli:

$$|y_1(x) - y_0| = \left| \int_{x_0}^x f(t, y_0(t)) dt \right| \leq M |x - x_0|.$$

Aytaylik, (1.13.9) tengsizlik biror $n \in N$ uchun bajarilsin. U holda (1.13.9) bahoni $n+1$ uchun bajarilishini ko'rsatamiz. Lipshtits shartidan foydalanib quyidagi ayirmani baholaymiz:

$$\begin{aligned} |y_{n+1}(x) - y_n(x)| &= \left| \int_{x_0}^x [f(t, y_n(t)) - f(t, y_{n-1}(t))] dt \right| \leq \\ &\leq \left| \int_{x_0}^x |f(t, y_n(t)) - f(t, y_{n-1}(t))| dt \right| \leq N \left| \int_{x_0}^x |y_n(t) - y_{n-1}(t)| dt \right| = \\ &= \frac{N^n M}{n!} \left| \int_{x_0}^x |t - x_0|^n dt \right| = MN^n \frac{|x - x_0|^{n+1}}{(n+1)!}. \end{aligned}$$

Agar $|x - x_0| \leq h$ deb, ushbu

$$a_n = \frac{MN^{n-1} h^n}{n!}$$

belgilashdan foydalansak, (1.13.9) baho quyidagi

$$|y_n(x) - y_{n-1}(x)| \leq MN^{n-1} \frac{h^n}{n!} \equiv a_n, \quad n \in N \quad (1.13.10)$$

ko'rinishni oladi. Avvalo $\{a_n\}$ ketma-ketlik ushbu

$$a_n \leq \frac{C}{2^n}, \quad C = const, \quad n \in N$$

tengsizlikni qanoatlantirishini ko'rsatamiz. Buning uchun $b_n = 2^n a_n$ ketma-ketlikni tuzib olamiz. Ko'rinib turibdiki,

$$\frac{b_{n+1}}{b_n} = \frac{2^{n+1} a_{n+1}}{2^n a_n} = \frac{2hN}{n+1} \rightarrow 0, \quad n \rightarrow \infty$$

munosabat o‘rinli. Bunga ko‘ra, shunday $n_0 \in N$ nomer topiladiki, $\frac{b_{n+1}}{b_n} < 1$ tengsizlik $n > n_0$ larda bajariladi. Bu esa $b_{n+1} < b_n$, ya’ni b_n ketma-ketlikning hadlari n_0 nomerdan boshlab kamayuvchi va chegaralangan ekanligini ko‘rsatadi. Demak, $b_n \leq C, \forall n > n_0$. Shuning uchun

$$a_n \leq \frac{C}{2^n}$$

tengsizlik bajariladi. Bundan va (1.13.10) tengsizlikdan

$$|y_n(x) - y_{n-1}(x)| \leq \frac{C}{2^n}, |x - x_0| \leq h \quad (1.13.11)$$

baho kelib chiqadi.

Endi $\{y_n(x)\}_{n=0}^{\infty}$, $x \in [x_0 - h, x_0 + h]$ ketma-ketlik Koshi kriteriyasini qanoatlantirishini ko‘rsatamiz. Buning uchun quyidagi ayirmani baholaymiz:

$$\begin{aligned} & |y_N(x) - y_{N+p}(x)| = |y_N(x) - y_{N+1}(x) + y_{N+1}(x) - y_{N+2}(x) + \\ & + y_{N+2}(x) - y_{N+3}(x) + \dots + y_{N+p-2}(x) - y_{N+p-1}(x) + \\ & + y_{N+p-1}(x) - y_{N+p}(x)| \leq \\ & \leq \sum_{k=1}^p |y_{N+k}(x) - y_{N+k-1}(x)| \leq \sum_{k=1}^p \frac{C}{2^{N+k}} = \frac{C}{2^N} \sum_{k=1}^p \frac{1}{2^k} \leq \frac{C}{2^N}. \end{aligned} \quad (1.13.12)$$

Bu tengsizlikdan

$$|y_N(x) - y_{N+p}(x)| \rightarrow 0, N \rightarrow \infty$$

ekanligi kelib chiqadi. Bu esa $\{y_n(x)\}$, $x \in [x_0 - h, x_0 + h]$ ketma-ketlikning fundamentalligini va Koshi kriteriyasiga asosan uning (1.13.8) ko‘rinishdagi chekli limitga ega ekanligini ko‘rsatadi.

Quyidagi

$$\sum_{k=1}^p \frac{1}{2^k} = \frac{1}{2} \frac{1 - \left(\frac{1}{2}\right)^p}{1 - \frac{1}{2}} = 1 - \left(\frac{1}{2}\right)^p$$

geometrik progressiya yig‘indisini topish formulasidan foydalanib, (1.13.12) tengsizlikni

$$|y_N(x) - y_{N+p}(x)| \leq \frac{C}{2^N} \left(1 - \left(\frac{1}{2}\right)^p\right)$$

ko‘rinishda yozish mumkin. Bu tengsizlikda $p \rightarrow \infty$ da limitga o‘tib, (1.13.8) munosabatni inobatga olsak,

$$|y_N(x) - y(x)| \leq \frac{C}{2^N}$$

kelib chiqadi. Barcha $x \in [x_0 - h, x_0 + h]$ larda (1.13.8) limitning mavjudligi $y(x)$ ning shu kesmada aniqlangan funksiya ekanligini bildiradi.

Endi $y(x)$ funksiyaning $[x_0 - h, x_0 + h]$ kesmada uzluksizligini va uning grafigi P to'g'ri to'rtburchakda yotishini ko'rsatamiz. Buning uchun $\forall x_1, x_2 \in [x_0 - h, x_0 + h]$ ikki nuqta olib ushbu $|y_n(x_1) - y_n(x_2)|$ ayirmani baholaymiz:

$$\begin{aligned} |y_n(x_1) - y_n(x_2)| &= \left| \int_{x_0}^{x_1} f(t, y_{n-1}(t)) dt - \int_{x_0}^{x_2} f(t, y_{n-1}(t)) dt \right| = \left| \int_{x_1}^{x_2} f(t, y_{n-1}(t)) dt \right| \leq \\ &\leq \left| \int_{x_1}^{x_2} |f(t, y_{n-1}(t))| dt \right| = M |x_1 - x_2|; \quad |y_n(x_1) - y_n(x_2)| \leq M |x_1 - x_2|. \end{aligned}$$

Bu oxirgi tengsizlikda $n \rightarrow \infty$ da limitga o'tsak,

$$|y(x_1) - y(x_2)| \leq M |x_1 - x_2|$$

baho kelib chiqadi. Bundan esa $y(x)$ funksiyaning $[x_0 - h, x_0 + h]$ kesmada uzluksizligi kelib chiqadi.

Yuqorida isbotlangan (1.13.7'), ya'ni

$$|y_n(x) - y_0| \leq b, \quad x \in [x_0 - h, x_0 + h]$$

tengsizlikda $n \rightarrow \infty$ da limitga o'tib,

$$|y(x) - y_0| \leq b, \quad x \in [x_0 - h, x_0 + h]$$

bahoni olamiz. Bu esa $y(x)$, $x \in [x_0 - h, x_0 + h]$ funksiyaning grafigi $(x, y(x)) \in P$ to'g'ri to'rtburchakda joylashishini ko'rsatadi.

Nihoyat $y(x)$, $x \in [x_0 - h, x_0 + h]$ uzluksiz funksiyaning (1.13.5) integral tenglamani qanoatlantirishini ko'rsatamiz. Avvalo Lipshtits shartidan foydalanib quyidagi ayirmani baholaymiz:

$$\left| \int_{x_0}^x [f(t, y_{n-1}(t)) - f(t, y(t))] dt \right| \leq N \left| \int_{x_0}^x |y_n(t) - y(t)| dt \right| = N \frac{C}{2^{n-1}} h \rightarrow 0, \quad n \rightarrow \infty.$$

Quyidagi

$$\begin{aligned} y_n(x) &= y_0 + \int_{x_0}^x f(t, y_{n-1}(t)) dt = \\ &= y_0 + \int_{x_0}^x f(t, y(t)) dt + \int_{x_0}^x [f(t, y_{n-1}(t)) - f(t, y(t))] dt \end{aligned}$$

tenglikda $n \rightarrow \infty$ da limitga o'tib, ushbu

$$y(x) = y_0 + \int_{x_0}^x f(t, y(t)) dt$$

integral tenglamani hosil qilamiz. Bu esa $y(x)$, $x \in [x_0 - h, x_0 + h]$ uzluksiz funksiya (1.13.5) integral tenglamaning yechimidan iborat ekanligini bildiradi. Shunday qilib, (1.13.1)-(1.13.2) Koshi masalasining $[x_0 - h, x_0 + h]$ kesmada aniqlangan $y(x)$ yechimi mavjud ekan.

Berilgan (1.13.1)-(1.13.2) Koshi masalasi yechimining yagonaligini ko'rsatish uchun quyidagi tasdiqdan foydalanamiz.

1.13.1-lemma (Gronuolla). Faraz qilaylik, $[x_0, x]$ kesmada $u(x)$, $v(x)$ funksiyalar uzluksiz va manfiy bo'lmasin. Agar ular uchun, ushbu

$$u(x) \leq A + \left| \int_{x_0}^x u(t)v(t)dt \right|, \quad A \geq 0 \quad (1.13.13)$$

baho o'rinli bo'lsa, u holda

$$u(x) \leq A \exp \left\{ \int_{x_0}^x v(t)dt \right\} \quad (1.13.14)$$

tengsizlik bajariladi.

Isbot. Aytaylik, $A > 0$, $x \geq x_0$ bo'lsin. U holda (1.13.13) tengsizlikda modul ishorasini tashlab va uni $v(x)$ ga ko'paytirsak,

$$\frac{u(x)v(x)}{A + \int_{x_0}^x u(t)v(t)dt} \leq v(x) \quad (1.13.15)$$

hosil bo'ladi. Oxirgi (1.13.15) tengsizlikni ushbu

$$\frac{d}{dx} \left(A + \int_{x_0}^x u(t)v(t)dt \right) = u(x)v(x)$$

munosabatdan foydalanib

$$\frac{d \left(A + \int_{x_0}^x u(t)v(t)dt \right)}{A + \int_{x_0}^x u(t)v(t)dt} \leq v(x)dx$$

ko'rinishda yozish mumkin. Bu tengsizlikning ikkala tomonini integrallab

$$\ln \left(A + \int_{x_0}^x u(t)v(t)dt \right) - \ln A \leq \int_{x_0}^x v(t)dt$$

munosabatni hosil qilamiz. Bundan

$$A + \int_{x_0}^x u(t)v(t)dt \leq A \exp \left\{ \int_{x_0}^x v(t)dt \right\} = A \exp \left\{ \left| \int_{x_0}^x v(t)dt \right| \right\}$$

kelib chiqadi. Lemma shartidagi (1.13.13) tengsizlikka asosan

$$u(x) \leq A + \left| \int_{x_0}^x u(t)v(t)dt \right| = A + \int_{x_0}^x u(t)v(t)dt \leq A \exp \left\{ \left| \int_{x_0}^x v(t)dt \right| \right\}$$

baho hosil bo'lad. Bu baho $A > 0, x < x_0$ larda ham o'rinli. Chunki $x < x_0$ larda (1.13.13) tengsizlikni quyidagi

$$u(x) \leq A - \int_{x_0}^x u(t)v(t)dt = A + \int_x^{x_0} u(t)v(t)dt$$

ko'rinishda yozish mumkin. Bundan ham

$$u(x) \leq A \exp \left\{ \left| \int_x^{x_0} v(t)dt \right| \right\} = A \exp \left\{ \left| \int_{x_0}^x v(t)dt \right| \right\}$$

kelib chiqadi.

Agar $A = 0$ bo'lsa, u holda $u(x) \equiv 0$ bo'lad. Haqiqatan ham

$$u(x) \leq \varepsilon + \left| \int_{x_0}^x u(t)v(t)dt \right|, \quad \forall \varepsilon > 0$$

bo'lsa, (1.13.14) dan

$$u(x) \leq \varepsilon \exp \left\{ \left| \int_{x_0}^x v(t)dt \right| \right\}$$

bahoga ega bo'lamiz. Bundan $\varepsilon \rightarrow +0$ da $u(x) \leq 0$ kelib chiqadi, bu esa $u(x) \geq 0$ shartga zid. Shuning uchun $u(x) \equiv 0$. ■

Yechimning yagonaligi. Aytaylik, $y_1(x), y_2(x)$ funksiyalar (1.13.1) differensial tenglamani va (1.13.2) boshlang'ich shartni qanoatlantirsin. Bundan tashqari ularning grafiklari P to'g'ri to'rtburchakda joylashsin, ammo

$$y_1(x) \neq y_2(x), \quad x \in [x_0 - h, x_0 + h]$$

bo'lsin. U holda ushbu

$$\begin{aligned} \frac{dy_1(x)}{dx} &= f(x, y_1(x)), \quad y_1(x_0) = y_0, \\ \frac{dy_2(x)}{dx} &= f(x, y_2(x)), \quad y_2(x_0) = y_0 \end{aligned}$$

tenglklardan, avvalo

$$y_1(x_0) - y_2(x_0) = 0,$$

soʻngra

$$\frac{d(y_1(x) - y_2(x))}{dx} = f(x, y_1(x)) - f(x, y_2(x))$$

munosabatni olamiz. Bu tenglikning ikki tomonini integrallab

$$y_1(x) - y_2(x) = \int_{x_0}^x [f(t, y_1(t)) - f(t, y_2(t))] dt$$

ifodani olamiz. Lipshtits shartidan foydalanib, oxirgi munosabatni baholaymiz:

$$|y_1(x) - y_2(x)| \leq \left| \int_{x_0}^x |f(t, y_1(t)) - f(t, y_2(t))| dt \right| \leq N \left| \int_{x_0}^x |y_1(t) - y_2(t)| dt \right|,$$

yaʼni

$$|y_1(x) - y_2(x)| \leq N \left| \int_{x_0}^x |y_1(t) - y_2(t)| dt \right|$$

bahoni olamiz. Ushbu

$$u(x) = |y_1(x) - y_2(x)| \geq 0, \quad v(x) = N > 0, \quad A = 0$$

belgilashlarni olib, Gronuolla tengsizligidan foydalansak, $u(x) \equiv 0, y_1(x) \equiv y_2(x)$ ekanligiga ishonch hosil qilamiz. Teorema toʻla isbot boʻldi. ■

Koʻpchilik hollarda (1.13.1)-(1.13.2) Koshi masalasining $y(x)$ yechimi bilan (1.13.6) tengliklar orqali aniqlangan $y_n(x)$, n -yaqinlashish orasidagi farqni hisoblashga toʻgʻri keladi. Buning uchun ushbu $|y(x) - y_n(x)|$ ayirmani baholashga toʻgʻri keladi. Avvalo biz $y_n(x)$ funksiyani quyidagi

$$y_n(x) = y_0 + \sum_{k=1}^n [y_k(x) - y_{k-1}(x)]$$

koʻrinishda yozib olamiz. Soʻngra bu tenglikning ikki tomonida $n \rightarrow \infty$ da limitga oʻtib

$$y(x) = y_0 + \sum_{k=1}^{\infty} [y_k(x) - y_{k-1}(x)]$$

munosabatni hosil qilamiz. Bundan va (1.13.10) tengsizlikdan foydalanib, quyidagi ayirmani baholaymiz:

$$\begin{aligned} |y(x) - y_n(x)| &= \left| y_0 + \sum_{k=1}^{\infty} [y_k(x) - y_{k-1}(x)] - y_n(x) \right| = \\ &= \left| \sum_{k=n+1}^{\infty} [y_k(x) - y_{k-1}(x)] \right| \leq \sum_{k=n+1}^{\infty} |y_k(x) - y_{k-1}(x)| \leq \end{aligned}$$

$$\leq \sum_{k=n+1}^{\infty} MN^{k-1} \frac{h^k}{k!} = Mh \frac{(Nh)^n}{(n+1)!} \sum_{j=0}^{\infty} \frac{(Nh)^j}{j!} = Mhe^{hN} \frac{(Nh)^n}{(n+1)!},$$

ya'ni

$$|y(x) - y_n(x)| \leq Mhe^{hN} \frac{(Nh)^n}{(n+1)!}. \quad (1.13.16)$$

Bu yerda

$$M = \max_{(x,y) \in P} |f(x,y)|, \quad |x - x_0| \leq h, \quad h = \min\left(a, \frac{b}{M}\right), \quad N - \text{Lipshits o'zgarmasi.}$$

Mustaqil yechish uchun mashqlar [21], §7, №221-223, 225, 226, 228.

14-§. Koshi masalasining korrektiligi

Ushbu

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0 \quad (1.14.0)$$

Koshi masalasini qaraylik.

1.14.1-ta'rif. Agar (1.14.0) masalasining yechimi

1) mavjud,

2) yagona,

3) berilganlarga nisbatan uzluksiz bog'liq (turg'un)

bo'lsa, Koshi masalasi korrekt deyiladi.

Berilgan (1.14.0) masalaning korrektiligini o'rganish uchun quyidagi

$$\frac{dy}{dx} = f_1(x, y), \quad y(x_0) = y_0^{(1)}, \quad (1.14.1)$$

$$\frac{dy}{dx} = f_2(x, y), \quad y(x_0) = y_0^{(2)} \quad (1.14.2)$$

Koshi masalalarini qaraymiz. Aytaylik, $y_j(x)$, $j=1,2$, $x \in [x_0 - h, x_0 + h]$ funksiyalar bu Koshi masalalarining yechimlaridan iborat bo'lsin. Bu yerda

$$h = \min\left(a; \frac{b}{M}\right), \quad M_j = \max_P |f_j(x, y)|, \quad j=1,2, \quad M = \max(M_1; M_2).$$

1.14.2-ta'rif. Agar $\forall \varepsilon > 0$ soni uchun $\exists \delta > 0$ soni topilib ushbu

$$|f_1(x, y) - f_2(x, y)| < \delta, \quad |y_0^{(1)} - y_0^{(2)}| < \delta \quad (1.14.3)$$

tengsizliklari bajarilganda

$$|y_1(x) - y_2(x)| < \varepsilon, \quad |x - x_0| \leq h \quad (1.14.4)$$

baho o'rinli bo'lsa, Koshi masalasi berilganlarga nisbatan uzluksiz bog'liq (turg'un) deyiladi.

1.14.1-teorema. Aytaylik $P = \{(x, y) \in R^2 : |x - x_0| \leq a, |y - y_0| \leq b\}$ sohada $f_1(x, y)$ va $f_2(x, y)$ funksiyalar uzluksiz bo‘lib, y o‘zgaruvchi bo‘yicha Lipshits shartini qanoatlantirsin. U holda

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

Koshi masalasi korrekt bo‘ladi.

Isbot. Oldingi paragrafda Koshi masalasi yechimining mavjudligi va yagonalagi ko‘rsatilgan edi. Endi Koshi masalasi yechimining berilganlarga nisbatan uzluksiz bog‘liq (turg‘un) ligini ko‘rsatamiz. Buning uchun quyidagi belgilashni kiritamiz:

$$\Delta = \max_{(x, y) \in P} |f_1(x, y) - f_2(x, y)|.$$

Berilgan (1.14.1), (1.14.2) Koshi masalalaridan

$$\frac{dy_1}{dx} - \frac{dy_2}{dx} = f_1(x, y) - f_2(x, y), \quad y_1(x_0) - y_2(x_0) = y_0^{(1)} - y_0^{(2)}$$

munosabatlarni topamiz va uni (x_0, x) oraliq bo‘yicha integrallab

$$\begin{aligned} y_1(x) - y_2(x) - (y_0^{(1)} - y_0^{(2)}) &= \int_{x_0}^x [f_1(t, y_1(t)) - f_2(t, y_2(t))] dt = \\ &= \int_{x_0}^x [f_1(t, y_1(t)) - f_2(t, y_1(t)) + f_2(t, y_1(t)) - f_2(t, y_2(t))] dt \end{aligned}$$

tenglikni hosil qilamiz. Bu yerda Lipshits shartidan va uchburchak tengsizligidan foydalanib quyidagi bahoni olamiz:

$$\begin{aligned} |y_1(x) - y_2(x)| &\leq |y_0^{(1)} - y_0^{(2)}| + \left| \int_{x_0}^x |f_1(t, y_1(t)) - f_2(t, y_1(t))| dt \right| + \\ &+ \left| \int_{x_0}^x |f_2(t, y_1(t)) - f_2(t, y_2(t))| dt \right| \leq |y_0^{(1)} - y_0^{(2)}| + \\ &+ \max_P |f_1(t, y_1(t)) - f_2(t, y_1(t))| \cdot |x - x_0| + N \left| \int_{x_0}^x |y_1(t) - y_2(t)| dt \right|. \end{aligned}$$

Agar $u(x) = |y_1(x) - y_2(x)|$, $v(x) = N > 0$, $A = |y_0^{(1)} - y_0^{(2)}| + h\Delta \geq 0$ belgilashdan va Gronuolla tengsizligidan foydalansak, oxirgi bahodan

$$|y_1(x) - y_2(x)| \leq (|y_0^{(1)} - y_0^{(2)}| + h\Delta) e^{N|x-x_0|} \leq (|y_0^{(1)} - y_0^{(2)}| + h\Delta) e^{Nh},$$

$$x \in [x_0 - h, x_0 + h] \tag{1.14.5}$$

kelib chiqadi. Bu yerda $\forall \varepsilon > 0$ sonini olib $\delta(\varepsilon) = \frac{\varepsilon}{1+h} e^{-Nh}$ deb tanlasak, oxirgi tengsizlikdan $\Delta < \delta$, $|y_0^1 - y_0^2| < \delta$ bo'lganda

$$|y_1(x) - y_2(x)| < \delta(1+h)e^{Nh} = \varepsilon, x \in [x_0 - h, x_0 + h]$$

baho kelib chiqadi. Bu esa Koshi masalasining turg'unligini ko'rsatadi. Bundan va yechimning mavjudligi hamda yagonaligidan uning korrektiligi kelib chiqadi. ■

Endi, ushbu

$$\frac{dy}{dx} = f(x, y), y(x_0) = y_0 \quad (1.14.6)$$

Koshi masalasining $y = y(x, x_0, y_0)$ yechimini boshlang'ich shartga uzluksiz bog'liqligini o'rganamiz. Buning uchun quyidagi Koshi masalasini ham qaraymiz:

$$\frac{dy}{dx} = f(x, y), y(x_0) = \bar{y}_0. \quad (1.14.7)$$

1.14.3-ta'rif. Agar $\forall \varepsilon > 0$ soni uchun shunday $\delta > 0$ soni topilib, $|y_0 - \bar{y}_0| < \delta$ tengsizligi bajarilganda

$$|y(x) - \bar{y}(x)| < \varepsilon, \forall x \in [x_0 - h, x_0 + h]$$

tengsizlik o'rinli bo'lsa, u holda (1.14.6) Koshi masalasining yechimi boshlang'ich shartga uzluksiz ravishda bog'liq deyiladi.

1.14.1-teoremada olingan (1.14.5) bahodan quyidagi natijalar kelib chiqadi.

1.14.1-natija. Agar $f(x, y)$ funksiya Koshi teoremasining shartlarini qanoatlantirsa, u holda (1.14.6) masalaning yechimi boshlang'ich shartga uzluksiz ravishda bog'liq bo'ladi.

Isbot. Qaralayotgan holda $\Delta = 0$ bo'lgani uchun (1.14.5) tengsizlik quyidagi

$$|y(x) - \bar{y}(x)| \leq |y_0 - \bar{y}_0| e^{Nh}, \forall x \in [x_0 - h, x_0 + h]$$

ko'rinishni oladi. Bunda $f_1 = f_2 = f$ ekanligi inobatga olindi.

Agar $\forall \varepsilon > 0$ sonini olib, $\delta(\varepsilon) > 0$ sonini $\delta(\varepsilon) = \varepsilon e^{-Nh}$ deb tanlasak, u holda $|y_0 - \bar{y}_0| < \delta$ tengsizligi bajarilganda

$$|y(x) - \bar{y}(x)| < \delta e^{Nh} = \varepsilon, \forall x \in [x_0 - h, x_0 + h]$$

baho o'rinli bo'ladi. Bu esa (1.14.6) Koshi masalasining yechimi boshlang'ich shartga uzluksiz bog'liq ekanini bildiradi. ■

Nihoyat (1.14.5) bahodan Koshi masalasi differensial tenglamaning o'ng tomoniga nisbatan uzluksiz bog'liq ekanligi ham kelib chiqadi.

Aytaylik, (1.14.1), (1.14.2) masalalarda $y_0^{(1)} = y_0^{(2)}$ bo'lsin, u holda

$$\frac{dy}{dx} = f_1(x, y), \quad y(x_0) = y_0,$$

$$\frac{dy}{dx} = f_2(x, y), \quad y(x_0) = y_0$$

munosabatlarga ega bo'lamiz.

1.14.2-natija. Agar $f_j(x, y)$, $j=1,2$ funksiyalar Koshi teoremasining shartlarini qanoatlantirsa, u holda Koshi masalasining yechimi differensial tenglamaning o'ng tomoniga uzluksiz ravishda bog'liq bo'ladi.

Isbot. Qaralayotgan holda $y_0^{(1)} = y_0^{(2)} = y_0$ bo'lgani uchun (1.14.5) tengsizlik quyidagi ko'rinishda yoziladi:

$$|y_1(x) - y_2(x)| \leq h\Delta e^{Nh}, \quad x \in [x_0 - h, x_0 + h].$$

Ixtiyoriy $\forall \varepsilon > 0$ sonini olib, $\delta(\varepsilon) > 0$ sonini $\delta(\varepsilon) = \frac{\varepsilon}{h} e^{-Nh}$ ko'rinishda tanlasak, $\Delta < \delta(\varepsilon)$ tengsizligi bajarilganda

$$|y_1(x) - y_2(x)| < \delta h e^{Nh} = \varepsilon, \quad x \in [x_0 - h, x_0 + h]$$

baho o'rinli bo'ladi. Bu esa Koshi masalasining yechimi differensial tenglamaning o'ng tomoniga uzluksiz ravishda bog'liq ekanligini bildiradi. ■

Demak, Koshi masalasi korrekt bo'lishi uchun uning yechimi boshlang'ich shartga va differensial tenglamaning o'ng tomoniga uzluksiz ravishda bog'liq bo'lishi kerak ekan.

15-§. Differensial tenglama yechimining silliqliigi

Ushbu

$$y' = f(x, y) \tag{1.15.1}$$

differensial tenglamani qaraylik.

1.15.1-teorema. Faraz qilaylik, $f(x, y)$ funksiya x, y o'zgaruvchilar bo'yicha $n \geq 1$ marta uzluksiz differensiullanuvchi bo'lsin. U holda (1.15.1) differensial tenglamaning ixtiyoriy yechimi $(n+1)$ marta differensiullanuvchi bo'ladi.

Isbot. Aytaylik, $n=1$ bo'lsin. U holda (1.15.1) differensial tenglamaning $y(x)$ yechimi uzluksiz differensiullanuvchi bo'ladi. Shuning uchun (1.15.1) tenglamani ushbu

$$\frac{dy}{dx} = f(x, y(x)) \tag{1.15.2}$$

ko'rinishda yozish mumkin. Teorema shartiga ko'ra $f(x, y)$ x, y o'zgaruvchilar bo'yicha differensiullanuvchi bo'lgani uchun $f(x, y(x))$ x o'zgaruvchining funksiyasi sifatida uzluksiz differensiullanuvchi bo'ladi (bunda murakkab funksiyaning differensiullanuvchiligi haqidagi teoreмага

asoslanildi). Bundan (1.15.2) ayniyatning o'ng tomoni uzluksiz differensiallanuvchi ekanligi, ya'ni y'' hosilaning mavjudligi kelib chiqadi.

Aytaylik, $n=2$ bo'lsin. U holda (1.15.2) ayniyatning ikki tomonini differensiallab

$$\frac{d^2 y(x)}{dx^2} = \frac{\partial f(x, y(x))}{\partial y} \cdot \frac{dy}{dx} \quad (1.15.3)$$

tenglikni hosil qilamiz. Bu ayniyatga yuqoridagi g'oyani qo'llab $y(x)$ yechimning $y'''(x)$ uchinchi tartibli hosilasining mavjudligini ko'rsatish mumkin. Shu jarayonni ketma-ket n marta qo'llash natijasida $y(x)$ yechimning $(n+1)$ marta differensiallanuvchi ekanligini ko'rsatish mumkin. ■

16-§. Differensial tenglama yechimining parametrlarga va boshlang'ich shartlarga bog'liqligi

Biror fizik jarayonni tavsiflovchi differensial tenglama parametrlarga (jumladan massa, elastiklik koeffitsiyentlari va hakoza fizik kattaliklar) bog'liq bo'ladi. Bu parametrlarning qiymatlarini real masalalarda aniq o'lchamini hisoblashning imkoni yo'q, odatda taqribiy hisoblanadi. Ma'lum jarayonni tavsiflovchi differensial tenglamani keltirib chiqarish jarayonida ham xatolikka yo'l qo'yiladi.

Shunday qilib differensial tenglama real jarayonni tavsiflashi uchun, uning yechimi parametrlarga uzluksiz ravishda bog'liq bo'lishi kerak, ya'ni parametrlarning kichik o'zgarishiga differensial tenglamaning yechimi ham mos ravishda kichik o'zgarishi lozim.

1.16.1-teorema. Agar $f(x, y, \lambda)$ funksiya

$$P = \{(x, y, \lambda) \in R^3 : |x - x_0| \leq a, |y - y_0| \leq b, |\lambda - \lambda_0| \leq c\}$$

sohada aniqlangan uzluksiz bo'lib, uzluksiz $f'_y(x, y, \lambda)$ va $f'_\lambda(x, y, \lambda)$ hosilalarga ega bo'lsa, u holda ushbu

$$y' = f(x, y, \lambda), \quad y(x_0, \lambda) = y_0 \quad (1.16.1)$$

Koshi masalasining $y = y(x, \lambda)$ yechimi uchun quyidagi tasdiqlar o'rinli:

1. (x, λ) -o'zgaruvchilarning uzluksiz funksiyasidan iborat bo'ladi.
2. $\frac{\partial y(x, \lambda)}{\partial \lambda} = u(x, \lambda)$ uzluksiz funksiya bo'lib,

$$\frac{du}{dx} = \frac{\partial f(x, y, \lambda)}{\partial y} u + \frac{\partial f(x, y, \lambda)}{\partial \lambda}, \quad u(x_0, \lambda) = 0$$

chiziqli differensial tenglamani qanoatlantiradi. Bunda

$$|x - x_0| \leq h, \quad h = \min\left(a, \frac{b}{M}\right), \quad M = \max_P |f(x, y, \lambda)|.$$

Isbot. 1. Ixtiyoriy $\forall \lambda_1, \lambda_2 \in [\lambda_0 - c, \lambda_0 + c]$ nuqtalarni olib, quyidagi

$$y' = f(x, y, \lambda_1), y(x_0, \lambda_1) = y_0 \quad (1.16.2)$$

$$y' = f(x, y, \lambda_2), y(x_0, \lambda_2) = y_0 \quad (1.16.3)$$

Koshi masalalarini qaraylik. Shu bilan bir qatorda, ularning yechimlarini mos ravishda $y(x, \lambda_1)$ va $y(x, \lambda_2)$ orqali belgilaylik.

Teorema shartiga ko'ra, $f'_y(x, y, \lambda)$ va $f'_\lambda(x, y, \lambda)$ funksiyalar P sohada uzluksiz bo'lganliklari uchun shunday $\exists N_1 > 0, N_2 > 0$ sonlari topilib,

$$|f'_y(x, y, \lambda)| \leq N_1, \quad |f'_\lambda(x, y, \lambda)| \leq N_2,$$

tengsizliklar o'rinli bo'ladi. Bu munosabatlardan foydalanib quyidagi

$$|f(x, y_1, \lambda) - f(x, y_2, \lambda)| = |f'_y(x, y_1 + \theta(y_2 - y_1), \lambda)| \cdot |y_1 - y_2| \leq N_1 |y_1 - y_2|,$$

$$0 < \theta < 1, \quad \forall (x, y_1, \lambda), (x, y_2, \lambda) \in P$$

$$|f(x, y, \lambda_1) - f(x, y, \lambda_2)| = |f'_\lambda(x, y, \lambda_1 + \theta(\lambda_2 - \lambda_1))| \cdot |\lambda_1 - \lambda_2| \leq N_2 |\lambda_1 - \lambda_2|,$$

$$\forall (x, y, \lambda_1), (x, y, \lambda_2) \in P$$

baholarni olamiz. Ushbu

$$y(x, \lambda_1) = y_0 + \int_{x_0}^x f(t, y(t, \lambda_1), \lambda_1) dt$$

$$y(x, \lambda_2) = y_0 + \int_{x_0}^x f(t, y(t, \lambda_2), \lambda_2) dt$$

integral tenglamalardan foydalanib

$$|y(x, \lambda_1) - y(x, \lambda_2)|$$

ayirmani baholaymiz:

$$\begin{aligned} |y(x, \lambda_1) - y(x, \lambda_2)| &= \left| \int_{x_0}^x f(t, y(t, \lambda_1), \lambda_1) dt - \int_{x_0}^x f(t, y(t, \lambda_2), \lambda_2) dt \right| \leq \\ &\leq \left| \int_{x_0}^x |f(t, y(t, \lambda_1), \lambda_1) - f(t, y(t, \lambda_2), \lambda_2)| dt \right| \leq \left| \int_{x_0}^x |f(t, y(t, \lambda_1), \lambda_1) - f(t, y(t, \lambda_2), \lambda_1)| dt \right| + \\ &+ \left| \int_{x_0}^x |f(t, y(t, \lambda_2), \lambda_1) - f(t, y(t, \lambda_2), \lambda_2)| dt \right| \leq N_1 \left| \int_{x_0}^x |y(t, \lambda_1) - y(t, \lambda_2)| dt \right| + \\ &+ N_2 |\lambda_1 - \lambda_2| \cdot |x - x_0| \leq N_2 |\lambda_1 - \lambda_2| \cdot h + N_1 \left| \int_{x_0}^x |y(t, \lambda_1) - y(t, \lambda_2)| dt \right|. \end{aligned}$$

Demak $|y(x, \lambda_1) - y(x, \lambda_2)|$ funksiya quyidagi

$$|y(x, \lambda_1) - y(x, \lambda_2)| \leq N_2 |\lambda_1 - \lambda_2| \cdot h + N_1 \left| \int_{x_0}^x |y(t, \lambda_1) - y(t, \lambda_2)| dt \right|$$

tengsizlikni qanoatlantirar ekan. Bunda, ushbu

$$u(x) = |y(x, \lambda_1) - y(x, \lambda_2)|, \quad v(x) = N_1, \quad A = N_2 \cdot |\lambda_1 - \lambda_2| \cdot h$$

belgilashni olib, Cronuolla tengsizligidan foydalansak

$$|y(x, \lambda_1) - y(x, \lambda_2)| \leq N_2 |\lambda_1 - \lambda_2| \cdot h e^{N_1 h}$$

baho hosil bo'ladi. Agar ixtiyoriy $\forall \varepsilon > 0$ soni uchun $\delta(\varepsilon) > 0$ sonini

$$\delta = \frac{\varepsilon e^{-N_1 h}}{N_2 h}$$

deb tanlasak, u holda $|\lambda_1 - \lambda_2| < \delta$ tengsizligi bajarilganda

$$|y(x, \lambda_1) - y(x, \lambda_2)| < N_2 \delta h e^{N_1 h} = \varepsilon, \quad \forall x \in [x_0 - h, x_0 + h], \quad \forall \lambda_1, \lambda_2 \in [\lambda_0 - c, \lambda_0 + c]$$

bahoning o'rinli bo'lishi kelib chiqadi. Bu esa $y(x, \lambda)$ yechimning (x, λ) o'zgaruvchilarga nisbatan uzluksiz ekanligini bildiradi. Teoremaning birinchi qismi isbotlandi.

2. Aytaylik $y = y(x, \lambda)$ funksiya (1.16.1) masalaning yechimi bo'lsin. U holda $y = y(x, \lambda + \Delta\lambda)$ funksiya ushbu

$$\frac{dy(x, \lambda + \Delta\lambda)}{dx} = f(x, y(x, \lambda + \Delta\lambda), \lambda + \Delta\lambda), \quad y(x_0, \lambda + \Delta\lambda) = y_0 \quad (1.16.4)$$

Koshi masalasining yechimi bo'ladi. $y = y(x, \lambda)$ -yechimning orttirmasi $\Delta y(x, \lambda) = y(x, \lambda + \Delta\lambda) - y(x, \lambda)$ bo'lgani uchun hamda

$$\frac{d\Delta y(x, \lambda)}{dx} = f(x, y(x, \lambda), \lambda), \quad y(x_0, \lambda) = y_0 \quad (1.16.5)$$

o'rinli ekanligini inobatga olib ushbu

$$\frac{d[y(x, \lambda + \Delta\lambda) - y(x, \lambda)]}{dx} = f(x, y(x, \lambda + \Delta\lambda), \lambda + \Delta\lambda) - f(x, y(x, \lambda), \lambda)$$

tenglikka ega bo'lamiz. Bu tenglikni

$$\frac{d\Delta y(x, \lambda)}{dx} = f(x, y(x, \lambda + \Delta\lambda), \lambda + \Delta\lambda) - f(x, y(x, \lambda), \lambda) \quad (1.16.6)$$

$$\Delta y(x_0, \lambda) = 0 \quad (1.16.6')$$

ko'rinishda yozish mumkin. Adamar lemmasiga (M.V. Fidaryuk "Обыкновенные дифференциальные уравнения" kitobining 106-108 betlari) ko'ra, (1.16.6) tenglamaning o'ng tomonini quyidagicha yozish mumkin:

$$\frac{d\Delta y(x, \lambda)}{dx} = F \Delta y(x, \lambda) + G \Delta \lambda,$$

ya'ni

$$\frac{d}{dx} \left(\frac{\Delta y(x, \lambda)}{\Delta \lambda} \right) = F \frac{\Delta y(x, \lambda)}{\Delta \lambda} + G. \quad (1.16.7)$$

Ushbu $y(x, \lambda)$ va $y(x, \lambda + \Delta \lambda)$ funksiyalar bir xil (bitta) boshlang'ich shartlarni qanoatlantirgani uchun

$$\left. \frac{\Delta y(x, \lambda)}{\Delta \lambda} \right|_{x=x_0} = 0 \quad (1.16.7')$$

shartga ega bo'lamiz. (1.16.7) tenglamaning o'ng tomoni $x, \Delta \lambda$ o'zgaruvchilar bo'yicha uzluksiz va $\frac{\Delta y}{\Delta \lambda}$ o'zgaruvchiga nisbatan uzluksiz differensiallanuvchi bo'lgani uchun Adamar lemmasiga asosan F, G funksiyalar ushbu $\frac{\partial f}{\partial y}, \frac{\partial f}{\partial \lambda}$ uzluksiz funksiyalarning integralidan iborat.

Yechimning parametrlarga uzluksiz bog'liqligidan $\frac{\Delta y}{\Delta \lambda}$ funksiya kichik $|\Delta \lambda|$ larda uzluksiz. Shuning uchun quyidagi chekli limit mavjud:

$$\lim_{\Delta \lambda \rightarrow 0} \frac{\Delta y(x, \lambda)}{\Delta \lambda} = \frac{\partial y(x, \lambda)}{\partial \lambda} \equiv u(x, \lambda).$$

Yana Adamar lemmasiga asosan

$$\lim_{\Delta \lambda \rightarrow 0} F = \frac{\partial f(x, y, \lambda)}{\partial y}, \quad \lim_{\Delta \lambda \rightarrow 0} G = \frac{\partial f(x, y, \lambda)}{\partial \lambda}$$

munosabatlarga ega bo'lamiz. Demak, $u(x, \lambda) = \frac{\partial y(x, \lambda)}{\partial \lambda}$ hosila quyidagi

$$\frac{du}{dx} = \frac{\partial f(x, y, \lambda)}{\partial y} u + \frac{\partial f(x, y, \lambda)}{\partial \lambda} \quad (1.16.8)$$

differensial tenglamani va

$$u(x_0, \lambda) = \left. \frac{\partial y(x, \lambda)}{\partial \lambda} \right|_{x=x_0} = 0 \quad (1.16.8')$$

boshlang'ich shartni qanoatlantirar ekan.

1.16.1-misol. Quyidagi

$$\frac{dy}{dx} = y^2 + 4\lambda x + \lambda^2, \quad y(1, \lambda) = 2\lambda - 1 \quad (1.16.9)$$

Koshi masalasi yechimining $y'_\lambda(x, \lambda)$ hosilasini $\lambda = 0$ nuqtadagi qiymatini toping.

Yechish. Bu holda (1.16.8) tenglama ushbu

$$\frac{du}{dx} = 2y \cdot u + 4x + 2\lambda, \quad u(1) = 2 \quad (1.16.10)$$

ko'rinishni oladi. Bu yerda $u(x, \lambda) = y'_\lambda(x, \lambda)$. Agar $\lambda = 0$ bo'lsa, (1.16.9) masala quyidagi

$$\frac{dy}{dx} = y^2, \quad y(1,0) = -1$$

ko‘rinishni oladi. Bu Koshi masalasini yechib $y = -\frac{1}{x}$ funksiyani topamiz.

Bundan foydalanib (1.16.10) masalani $\lambda = 0$ da

$$\frac{du}{dx} = -\frac{2}{x}u + 4x, \quad u(1) = 2$$

ko‘rinishda yozish mumkin. Hosil bo‘lgan chiziqli tenglamani yechib

$$u = x^2 + x^{-2},$$

ya’ni

$$y'_\lambda(x, \lambda)|_{\lambda=0} = x^2 + x^{-2}$$

topamiz.

1.16.2-Misol. Quyidagi

$$y' = \lambda(1-x) + y - y^2, \quad y(0, \lambda) = 0 \quad (1.16.11)$$

Koshi masalasi yechimining $y'_\lambda(x, \lambda)$ hosilasini $\lambda = 0$ nuqtadagi qiymatini toping.

Yechish. Qaralayotgan holda (1.16.8) tenglama

$$\frac{du}{dx} = [1 - 2y]u + 1 - x, \quad u|_{x=0} = 0 \quad (1.16.12)$$

ko‘rinishni oladi. $\lambda = 0$ holda (1.16.11) masala

$$y' = y - y^2, \quad y(0) = 0$$

ko‘rinishda bo‘lgani uchun $y = y(x, 0) = 0$ bo‘ladi. Bundan foydalanib (1.16.12) tenglama

$$\frac{du(x, 0)}{dx} = u(x, 0) + 1 - x, \quad u|_{x=0} = 0$$

ko‘rinishga keladi. Chiziqli tenglamani yechib

$$u(x, 0) = x, \quad \text{ya'ni} \quad y'_\lambda(x, 0) = u(x, 0) = x$$

ekanligini topamiz.

Endi, ushbu

$$y' = f(x, y), \quad y(x_0) = y_0 \quad (1.16.13)$$

Koshi masalasining $y = \varphi(x, x_0, y_0)$ yechimini x_0, y_0 boshlang‘ich shartga nisbatan silliqqligini o‘rganamiz.

1.16.2-teorema. Aytaylik $f(x, y)$ va $f'_y(x, y)$ funksiyalar

$$G = \{(x, y) \in R^2 : |x - x_0| \leq a, |y - y_0| \leq b\}$$

sohada uzluksiz bo‘lsin. U holda, shunday $h > 0$ soni topilib, (1.16.13) Koshi masalasining ushbu $[x_0 - h, x_0 + h]$ oraliqda aniqlangan $y = \varphi(x, x_0, y_0)$ yechimi uchun quyidagi tasdiqlar o‘rinli:

1. $\frac{\partial \varphi(x, x_0, y_0)}{\partial x_0}, \frac{\partial \varphi(x, x_0, y_0)}{\partial y_0}$ – xususiy hosilalar uzluksiz funksiyalardan

iborat bo‘lib, mos ravishda ushbu

$$\frac{du}{dx} = \frac{\partial f(x, \varphi(x, x_0, y_0))}{\partial y} \cdot u, \quad u|_{x=x_0} = 0, \quad (1.16.14)$$

$$\frac{dv}{dx} = \frac{\partial f(x, \varphi(x, x_0, y_0))}{\partial y} \cdot v, \quad v|_{x=x_0} = 1$$

tenglamalarni qanoatlantiradi. Bu yerda

$$u = u(x, x_0, y_0) \equiv \frac{\partial \varphi(x, x_0, y_0)}{\partial x_0}, \quad (1.16.15)$$

$$v = v(x, x_0, y_0) \equiv \frac{\partial \varphi(x, x_0, y_0)}{\partial y_0}$$

2. $\varphi''_{xx_0}, \varphi''_{xy_0}$ – aralash hosilalar uzluksiz.

Isbot. Avvalo ushbu

$$\frac{\partial y}{\partial x_0} = \frac{\partial \varphi(x, x_0, y_0)}{\partial x_0} = \frac{\partial y(x, x_0, y_0)}{\partial x_0}$$

xususiy hosilani mavjudligini ko‘rsatamiz. Buning uchun quyidagi yordamchi

$$y' = f(x, y), \quad y(x_0) = y_1 \quad (1.16.16)$$

Koshi masalasini ham qaraymiz. Berilgan (1.16.13) Koshi masalasining $y = y(x, x_0, y_0)$ yechimi $I = [x_0 - h, x_0 + h]$ oraliqda mavjud. Shu bilan bir qatorda (1.16.16) Koshi masalasining yechimi $y = y(x, x_0, y_1)$, $x \in [x_0 - h_1, x_0 + h_1] = I_1$ oraliqda mavjud. Bu yechimlar boshlang‘ich shartlarga nisbatan uzluksiz bo‘lgani uchun, ushbu

$$|y(x, x_0, y_0) - y(x, x_0, y_1)| \leq |y_0 - y_1| e^{L|x-x_0|}, \quad x \in I \cap I_1$$

baho o‘rinli, ya‘ni $|y_0 - y_1| \rightarrow 0$ da $|y(x, x_0, y_0) - y(x, x_0, y_1)| \rightarrow 0$ munosabat o‘rinli bo‘ladi. Bundan tashqari (1.16.13) va (1.16.16) Koshi masalalari quyidagi

$$y(x, x_0, y_0) = y_0 + \int_{x_0}^x f(t, y(t, x_0, y_0)) dt \quad (1.16.16')$$

$$y(x, x_0, y_1) = y_1 + \int_{x_0}^x f(t, y(t, x_0, y_1)) dt$$

integral tenglamalarga ekvivalent. Shu bilan bir qatorda $z(x)$ funksiyaga nisbatan ushbu

$$z(x) = 1 + \int_{x_0}^x \frac{\partial f(t, y(t, x_0, y_0))}{\partial y} z(t) dt$$

integral tenglamani ham qaraylik. Yuqoridagi integral tenglamalardan foydalanib quyidagi ayirmani hisoblaymiz:

$$\begin{aligned} & y(x, x_0, y_0) - y(x, x_0, y_1) - z(x)(y_0 - y_1) = \\ &= (y_0 - y_1) + \int_{x_0}^x [f(t, y(t, x_0, y_0)) - f(t, y(t, x_0, y_1))] dt - \\ & \quad - (y_0 - y_1) - \int_{x_0}^x \frac{\partial f(t, y(t, x_0, y_0))}{\partial y} z(t) dt (y_0 - y_1) = \\ &= \int_{x_0}^x \left[f(t, y(t, x_0, y_0)) - f(t, y(t, x_0, y_1)) - \frac{\partial f(t, y(t, x_0, y_0))}{\partial y} z(t)(y_0 - y_1) \right] dt = \\ &= \int_{x_0}^x \frac{\partial f(t, y(t, x_0, y_0))}{\partial y} (y(t, x_0, y_0) - y(t, x_0, y_1)) + \alpha(y(x, x_0, y_0), y(x, x_0, y_1)) dt - \\ & \quad - \int_{x_0}^x \frac{\partial f(t, y(t, x_0, y_0))}{\partial y} z(t)(y_0 - y_1) dt, \end{aligned}$$

ya'ni

$$\begin{aligned} & y(x, x_0, y_0) - y(x, x_0, y_1) - z(x)(y_0 - y_1) = \\ &= \int_{x_0}^x \frac{\partial f(t, y(t, x_0, y_0))}{\partial y} [y(t, x_0, y_0) - y(t, x_0, y_1) - z(t)(y_0 - y_1)] dt + \\ & \quad + \int_{x_0}^x \alpha [y(t, x_0, y_0), y(t, x_0, y_1)] dt. \end{aligned} \quad (1.16.17)$$

Bu yerda α cheksiz kichik miqdor, ya'ni $\alpha [y(t, x_0, y_0), y(t, x_0, y_1)] \rightarrow 0$ munosabat o'rinli bo'ladi, qachonki $|y(x, x_0, y_0) - y(x, x_0, y_1)| \rightarrow 0$ bo'lsa, bu esa $|y_0 - y_1| \rightarrow 0$ da o'rinli. Oxirgi (1.16.17) tenglikni $|f'_y(x, y(x, x_0, y_0))| \leq L, L > 0$ tengsizlikdan foydalanib, baholaymiz:

$$\begin{aligned} & |y(x, x_0, y_0) - y(x, x_0, y_1) - z(x)(y_0 - y_1)| \leq \\ & \leq L \left| \int_{x_0}^x |y(t, x_0, y_0) - y(t, x_0, y_1) - z(t)(y_0 - y_1)| dt \right| + \bar{0}(|y_0 - y_1|). \end{aligned} \quad (1.16.18)$$

Bu yerda $|y_0 - y_1| \rightarrow 0$ da yechimning boshlangich shartga uzluksiz bog'liqligidan $y(x, x_0, y_0) - y(x, x_0, y_1) \rightarrow 0$

bo'lishi, bundan esa o'z navbatida

$$\alpha[y(t, x_0, y_0), y(t, x_0, y_1)] \rightarrow 0$$

kelib chiqadi. Oxirgi (1.16.18) munosabatga Gronuolla tengsizligini qo'llasak quyidagi baho kelib chiqadi:

$$u(x) = |y(t, x_0, y_0) - y(t, x_0, y_1) - z(t)(y_0 - y_1)|$$

$$v(x) = L, \quad A = \bar{o}(|y_0 - y_1|)$$

$$|y(t, x_0, y_0) - y(t, x_0, y_1) - z(t)(y_0 - y_1)| \leq \bar{o}(|y_0 - y_1|) e^{L|x-x_0|}$$

Bu bahodan $|y_0 - y_1| \rightarrow 0$ da

$$|y(t, x_0, y_0) - y(t, x_0, y_1) - z(t)(y_0 - y_1)| \rightarrow 0$$

bo'lishi kelib chiqadi. Bu esa, o'z navbatida

$$\frac{|y(t, x_0, y_0) - y(t, x_0, y_1)|}{y_0 - y_1} = z(x) + \bar{o}(1)$$

ekanligini bildiradi. Bundan, ushbu $\frac{\partial y(x, x_0, y_0)}{\partial y_0}$ xususiy hosilaning

mavjudligi va

$$z(x) = \frac{\partial y(x, x_0, y_0)}{\partial y_0} = \frac{\partial \varphi(x, x_0, y_0)}{\partial y_0}$$

tenglik kelib chiqadi. Teoremaning qolgan bandlari ham xuddi shunday isbotlanadi.

1.16.3-misol. Ushbu

$$y' = \lambda y^2 + 1, \quad y(0) = 0$$

Koshi masalasi yechimining $z = y'_\lambda|_{\lambda=0}$ qiymatini toping.

Yechish. Avvalo $\lambda = 0$ bo'lgan holda

$$y' = 1, \quad y(0) = 0$$

masalaning $y(x, 0) = x$ yechimini topamiz. So'ngra $f(x, y, \lambda) = \lambda y^2 + 1$ tenglikdan

$$\frac{\partial f}{\partial y} = 2\lambda y, \quad \frac{\partial f}{\partial \lambda} = y^2$$

munosabatlarni aniqlaymiz. Endi yuqoridagi munosabatlardan foydalanib quyidagi

$$\frac{dz(x, \lambda)}{dx} = \frac{\partial f}{\partial y} z(x, \lambda) + \frac{\partial f}{\partial \lambda},$$

$$z(0, \lambda) = 0$$

differensial tenglamani tuzib olamiz:

$$\begin{cases} z'(x, \lambda) = 2\lambda yz + y^2, \\ z(0, \lambda) = 0. \end{cases}$$

Bu tenglamada y o'rniga $y = y(x, 0) = x$ ni qo'yib quyidagi

$$\begin{cases} z'(x, \lambda) = 2\lambda xz + x^2, \\ z(0, \lambda) = 0 \end{cases}$$

masalani hosil qilamiz. Bu yerda $\lambda = 0$ deb ushbu

$$\begin{cases} z'(x, 0) = x^2, \\ z(0, 0) = 0 \end{cases}$$

masalaning yechimini topamiz:

$$z(x, 0) = \frac{x^3}{3}.$$

Bu funksiya $y'_\lambda(x, \lambda)|_{\lambda=0} = z(x, 0) = \frac{x^3}{3}$ biz izlayotgan qiymatni beradi.

Ikkinchi tomondan berilgan tenglama o'zgaruvchilarga ajraladigan differensial tenglama bo'lgani uchun, uning yechimini topish mumkin:

$$\frac{dy}{(\sqrt{\lambda}y)^2 + 1} = dx, \quad \frac{1}{\sqrt{\lambda}} \arctg \sqrt{\lambda}y = x + c,$$

$$y(0) = 0 \rightarrow \frac{1}{\sqrt{\lambda}} \arctg \sqrt{\lambda}y = x, \quad y = \frac{1}{\sqrt{\lambda}} \operatorname{tg} \sqrt{\lambda}x.$$

Agar $y = y(x, \lambda)$ ni quyidagi

$$y(x, \lambda) = y(x, 0) + y'_\lambda(x, \lambda)|_{\lambda=0} \cdot \lambda + \bar{o}(\lambda)$$

ko'rinishda yozsak, u holda

$$y(x, \lambda) = \frac{1}{\sqrt{\lambda}} \operatorname{tg}(\sqrt{\lambda}x) = x + \lambda \frac{x^3}{3} + \bar{o}(\lambda)$$

hosil bo'ladi.

Yuqoridagi 1.16.1-teoremada bayon qilingan tasdiqni quyidagicha umumlashtirish mumkin.

1.16.3-teorema. Agar $f(x, y, \lambda)$ funksiya P sohada Koshi teoremasining shartlarini qanoatlantirib, x, y, λ o'zgaruvchilar bo'yicha $m \geq 2$ marta differensiallanuvchi bo'lsa, u holda (1.16.1) masalaning $y(x, \lambda)$ yechimi x, λ o'zgaruvchilar bo'yicha differensiallanuvchi bo'lib, λ o'zgaruvchi bo'yicha m marta differensiallanuvchi bo'ladi. Bundan tashqari $y(x, \lambda)$ yechimni λ^m ning darajalari bo'yicha Teylor formulasiga yoyish mumkin:

$$y(x, \lambda) = u_0(x) + \lambda u_1(x) + \lambda^2 u_2(x) + \dots + \lambda^m u_m(x) + \bar{o}(\lambda^m) \quad (1.16.19)$$

1.16.4-misol. Ushbu

$$y'(x) = \frac{x}{y} - 2\lambda x^2, \quad y(1) = 1 \quad (1.16.20)$$

masala $y(x, \lambda)$ yechimining λ bo'yicha yoyilmasini λ^2 gacha aniqlikda toping.

Yechish. Berilgan tenglama o'ng tomoni

$$f(x, y, \lambda) = \frac{x}{y} - 2\lambda x^2,$$

$y > 0$ sohada barcha tartibli hosilalarga ega. $\lambda = 0$ da berilgan masala ushbu

$$y' = \frac{x}{y}, \quad y(1) = 1$$

ko'rinishni oladi. Bu masala $y(x, 0) = x$ yechimga ega. Berilgan masalaning yechimini

$$y(x, \lambda) = u_0(x) + \lambda u_1(x) + \lambda^2 u_2(x) + \bar{o}(\lambda^2)$$

ko'rinishda izlaymiz. Bu yerda $u_0(x) = y(x, 0) = x$. Shuning uchun yuqoridagi yoyilma ushbu

$$y(x, \lambda) = x + \lambda u_1(x) + \lambda^2 u_2(x) + \bar{o}(\lambda^2) \quad (1.16.21)$$

ko'rinishni oladi. (1.16.21) yoyilmani (1.16.20) tenglamaga qo'yamiz:

$$1 + \lambda u_1' + \lambda^2 u_2' + \dots = \frac{x}{x + \lambda u_1 + \lambda^2 u_2 + \dots} - 2\lambda x^2,$$

$$y(1, \lambda) = 1 \rightarrow 1 + \lambda u_1(1) + \lambda^2 u_2(1) + \dots = 1 \Rightarrow$$

$$u_1(1) = 0, u_2(1) = 0, \dots$$

Endi ushbu

$$\begin{aligned} \frac{x}{x + \lambda u_1 + \lambda^2 u_2 + \dots} &= \frac{1}{1 + \lambda x^{-1} u_1 + \lambda^2 x^{-1} u_2 + \dots} = \\ &= 1 - \left(\frac{\lambda}{x} u_1 + \frac{\lambda^2}{x} u_2 + \dots \right) + \left(\frac{\lambda}{x} u_1 + \dots \right)^2 - \dots = \\ &= 1 - \frac{\lambda}{x} u_1 - \frac{\lambda^2}{x} u_2 + \frac{\lambda^2}{x^2} u_1^2 + \dots \end{aligned}$$

yoyilmadan foydalanib,

$$1 + \lambda u_1' + \lambda^2 u_2' + \dots = 1 - \frac{\lambda}{x} u_1 - \frac{\lambda^2}{x} u_2 + \frac{\lambda^2}{x^2} u_1^2 - 2\lambda x^2 + \dots$$

tenglikni hosil qilamiz. Bundan ushbu

$$u_1'(x) = -\frac{u_1(x)}{x} - 2x^2, \quad u_1(1) = 0, \quad (1.16.22)$$

$$u_2'(x) = -\frac{u_2(x)}{x} + \frac{u_1^2(x)}{x^2}, \quad u_2(1) = 0, \quad (1.16.23)$$

Koshi masalalarini topamiz. Avvalo (1.16.22) tenglamaning bir jinsli qismining umumiy yechimini topamiz:

$$1) \frac{du_1}{dx} = -\frac{u_1}{x} \Rightarrow \frac{du_1}{u_1} = -\frac{dx}{x},$$

$$\ln u_1 = -\ln x + \ln c; u_1(x) = \frac{c}{x}.$$

Soʻngra (1.16.22) tenglamaning xususiy yechimini topamiz:

$$\bar{u}_1(x) = ax^3; u_1'(x) = 3ax^2,$$

$$3ax^2 = -\frac{ax^3}{x} - 2x^2,$$

$$3ax^2 = -ax^2 - 2x^2,$$

$$3a = -a - 2, 4a = -2 \quad a = -\frac{1}{2}.$$

Demak, biz izlagan xususiy yechim quyidagi

$$\bar{u}_1(x) = -\frac{x^3}{2}$$

koʻrinishda boʻlar ekan. Bundan va boshlangʻich shartdan foydalanib (1.16.22) Koshi masalasining yechimini topamiz:

$$u_1(x) = \frac{c}{x} - \frac{x^3}{2}; \quad u(1) = 0,$$

$$c - \frac{1}{2} = 0, \quad c = \frac{1}{2},$$

$$u_1(x) = \frac{1}{2x} - \frac{x^3}{2}.$$

Endi quyidagi

$$\frac{du_2}{dx} = -\frac{u_2}{x} + \frac{\left(\frac{1}{2x} - \frac{x^3}{2}\right)^2}{x^2}, \quad u_2(1) = 0$$

masalaning yechimini topish bilan shugʻullanamiz. Bu tenglamani yechish uchun, avvalo uning bir jinsli qismini yechamiz:

$$\frac{du_2}{dx} = -\frac{u_2}{x} \Rightarrow u_2(x) = \frac{c}{x}.$$

Endi bir jinsli boʻlmagan differensial tenglamaning yechimini Lagranj usulidan foydalanib topamiz:

$$u_2(x) = \frac{c(x)}{x}, \quad u_2'(x) = \frac{c'(x)}{x} - \frac{1}{x^2}c(x),$$

$$\frac{c'(x)}{x} - \frac{1}{x^2}c(x) = -\frac{1}{x^2}c(x) + \frac{\left(\frac{1}{2x} - \frac{x^3}{2}\right)^2}{x^2},$$

$$c'(x) = \frac{\left(\frac{1}{2x} - \frac{x^3}{2}\right)^2}{x} = \frac{1}{x} \left(\frac{1}{4x^2} - \frac{1}{2}x^2 + \frac{x^6}{4} \right) = \frac{1}{4x^3} - \frac{1}{2}x + \frac{x^5}{4},$$

$$\frac{dc(x)}{dx} = \frac{1}{4x^3} - \frac{1}{2}x + \frac{x^5}{4},$$

$$c(x) = -\frac{1}{8x^2} - \frac{1}{4}x^2 + \frac{x^6}{24} + c_1,$$

$$u_2(x) = \frac{1}{x} \left(c_1 - \frac{1}{8x^2} - \frac{1}{4}x^2 + \frac{x^6}{24} \right),$$

$$u_2(1) = 0 \Rightarrow c_1 - \frac{1}{8} - \frac{1}{4} + \frac{1}{24} = 0,$$

$$c_1 - \frac{1}{3} = 0, \quad c_1 = \frac{1}{3},$$

$$u_2(x) = \frac{1}{x} \left(\frac{1}{3} - \frac{1}{8x^2} - \frac{1}{4}x^2 + \frac{x^6}{24} \right).$$

Demak, (1.16.21) formulaga asosan berilgan (1.16.20) masalaning $y(x, \lambda)$ yechimi uchun quyidagi

$$y(x, \lambda) = x + \lambda \left(\frac{1}{2x} - \frac{x^3}{2} \right) + \lambda^2 \left(\frac{1}{3x} - \frac{1}{8x^3} - \frac{1}{4}x + \frac{x^5}{24} \right) + o(\lambda^2)$$

asimptotik yoyilma o'rinli bo'lar ekan.

Mustaqil yechish uchun mashqlar [21], §13, №723-725.

17-§. Kichik parametrlar usuli

1. Quyidagi

$$\frac{dy}{dx} = f(x, y, \varepsilon), \quad y(x_0) = y_0 \quad (1.17.1)$$

Koshi masalasining yechimini $y(x, \varepsilon)$ orqali belgilaylik.

Aytaylik, ushbu ($\varepsilon = 0$)

$$\frac{dy}{dx} = f(x, y, 0), \quad y|_{x=x_0} = y_0 \quad (1.17.2)$$

Koshi masalasining $y = y_0(x)$, $x \in I = [0, l]$ yechimi mavjud va yagona bo'lsin. U holda quyidagi tasdiq o'rinli.

1.17.1-teorema. Agar $\varepsilon_0 > 0$ yetarli kichik son bo'lsa, u holda $|\varepsilon| \leq \varepsilon_0$ tengsizlikni qanoatlantiruvchi barcha ε lar uchun (1.17.1) masalaning $y(x, \varepsilon)$, $x \in I = [0, l]$ yechimi mavjud bo'lib, ixtiyoriy $\forall N \geq 1$ larda quyidagi

$$y(x, \varepsilon) = y_0(x) + \varepsilon y_1(x) + \dots + \varepsilon^{N-1} y_{N-1}(x) + R_N(x, \varepsilon) \quad (1.17.3)$$

yoyilma o‘rinli bo‘ladi. Bu yerda qoldiq had ushbu

$$|R_N(x, \varepsilon)| \leq c_N \varepsilon^N, \quad x \in I, \quad \varepsilon \leq \varepsilon_0 \quad (1.17.4)$$

tengsizlikni qanoatlantiradi. c_N —o‘zgarmas soni x, ε larga bog‘liq emas.

Isbot. Ushbu $y_0(x)$ funksiyani ma‘lum deb qaraymiz. U holda (1.17.3) yoyilmani qolgan hadlarini qanday aniqlash mumkinligini ko‘rsatamiz. Buning uchun (1.17.3) yoyilmani (1.17.1) differensial tenglamaga qo‘yib

$$\sum_{j=0}^{N-1} \varepsilon^j \frac{dy_j(x)}{dx} + O(\varepsilon^N) = f \left(x, \sum_{j=0}^{N-1} \varepsilon^j y_j(x) + O(\varepsilon^N), \varepsilon \right) \quad (1.17.5)$$

munosabatni hosil qilamiz. Bu tenglikning o‘ng tomonidagi funksiyani ε parametrning darajalari bo‘yicha $O(\varepsilon^N)$ aniqlikgacha Teylor formulasiga yoysak, quyidagi

$$f(x, \sum_{j=0}^{N-1} y_j(x) \varepsilon^j + \bar{o}(\varepsilon^N), \varepsilon) = f(x, y_0(x), 0) + \frac{\partial f(x, y_0, 0)}{\partial y} y_1 \varepsilon + \frac{\partial f(x, y_0, 0)}{\partial \varepsilon} \varepsilon + \dots$$

$$+ \frac{1}{(N-1)!} (\varepsilon y_1 \frac{\partial}{\partial y} + \varepsilon \frac{\partial}{\partial \varepsilon})^{N-1} f(x, y_0(x), 0) + \bar{o}(\varepsilon^N)$$

munosabat hosil bo‘ladi. Bundan, avvalo $y_0(x)$ ga nisbatan (1.17.2) Koshi masalasini hosil qilamiz. So‘ngra $y_1(x)$ —uchun ushbu

$$\frac{dy_1(x)}{dx} = \frac{\partial f(x, y_0(x), 0)}{\partial y} y_1(x) + \frac{\partial f(x, y_0(x), 0)}{\partial \varepsilon}, \quad y_1(0) = 0 \quad (1.17.6)$$

Koshi masalasiga ega bo‘lamiz. Bu esa birinchi tartibli chiziqli differensial tenglamadir. Shuning uchun uning yechimi $I = [0, l]$ oraliqda mavjud va yagona. Qolgan barcha $y_2(x), y_3(x), \dots$ hadlar uchun ham quyidagi

$$\frac{dy_n(x)}{dx} = \frac{\partial f(x, y_0(x), 0)}{\partial y} y_n(x) + F_n(x, y_0(x), \dots, y_{N-1}(x)), \quad y_n(0) = 0 \quad (1.17.7)$$

chiziqli differensial tenglamaga ega bo‘lamiz. Bunda F_n —ma‘lum funksiyalar. Bu differensial tenglamalar bir-biridan faqat o‘ng tomoniga farq qiladi. (1.17.7) ko‘rinishdagi Koshi masalalarining har birining $x \in I = [0, l]$ oraliqda aniqlangan yechimi mavjud va yagona bo‘lib, u $y_n(x) \in C^\infty(I)$ —cheksiz differensiallanuvchi funksiyadan iborat bo‘ladi.

1.17.2-teorema. Faraz qilaylik:

1. $f(x, y; \varepsilon)$ va $f'_y(x, y, \varepsilon)$ funksiyalar $G = \{0 \leq x \leq l, |y| \leq b, |\varepsilon| < \varepsilon_0\}$ sohada uzluksiz va tekis chegaralangan, ya‘ni

$$|f(x, y; \varepsilon)| \leq M, \quad |f'_y(x, y; \varepsilon)| \leq L$$

bo‘lsin.

2. Ushbu

$$\frac{dy}{dx} = f(x, y, 0), \quad y(0) = y_0$$

Koshi masalasining $y_0(x)$, $x \in [0, l]$ yechimi mavjud va yagona bo'lib, $D = \{(x, y) \in R^2 : 0 \leq x \leq l, |y| < b\}$ sohaga tegishli bo'lsin. U holda, har bir yetarli kichik ε lar uchun

$$\frac{dy}{dx} = f(x, y, \varepsilon), \quad y(0, \varepsilon) = y_0$$

Koshi masalasining $x \in [0, l]$ oraliqda aniqlangan yagona $y(x, \varepsilon) = y_\varepsilon(x)$ yechimi mavjud va u G sohaga tegishli bo'ladi hamda

$$\lim_{\varepsilon \rightarrow 0} y_\varepsilon(x) = y_0(x), \quad x \in [0, l]$$

munosabat x ga nisbatan tekis bajariladi.

Isbot. Quyidagi

$$Z_\varepsilon(x) = y_\varepsilon(x) - y_0(x)$$

belgilashdan foydalansak, ushbu

$$\frac{dZ_\varepsilon(x)}{dx} = f(x, Z_\varepsilon(x) + y_0(x), \varepsilon) - f(x, y_0(x))$$

$$Z_\varepsilon(0) = 0$$

Koshi masalasiga ega bo'lamiz. Bu masalaga ekvivalent bo'lgan integral tenglama tuzamiz:

$$Z_\varepsilon(x) = \int_0^x [f(t, Z_\varepsilon(t) + y_0(t), \varepsilon) - f(t, y_0(t), 0)] dt.$$

Oxirgi tenglikning o'ng tomonini quyidagi ko'rinishda yozish mumkin:

$$Z_\varepsilon(x) = \int_0^x [f(t, Z_\varepsilon(t) + y_0(t), \varepsilon) - f(t, y_0(t), \varepsilon)] dt + F(x, \varepsilon).$$

Bunda

$$F(x, \varepsilon) = \int_0^x [f(t, y_0(t), \varepsilon) - f(t, y_0(t), 0)] dt$$

ko'rinishda bo'lib, ushbu

$$|F(x, \varepsilon)| \leq \alpha(\varepsilon)$$

tengsizlikni qanoatlantiradi, chunki $f(x, y, \varepsilon) - \varepsilon$ parametr ga nisbatan uzluksiz. Bu yerda $\alpha(\varepsilon)$ - cheksiz kichik miqdor, ya'ni

$$\lim_{\varepsilon \rightarrow 0} F(x, \varepsilon) = 0.$$

Endi $Z_\varepsilon(x)$ funksiyani baholaymiz:

$$Z_\varepsilon(x) \leq \left| \int_0^x [f(t, z_\varepsilon(t) + y_0(t), \varepsilon) - f(t, y_0(t), \varepsilon)] dt \right| + |F(x, \varepsilon)| \leq L \int_0^x |Z_\varepsilon(t)| dt + \alpha(x).$$

Bu tengsizlikka Gronuolla lemmasini qo'llasak,

$$|Z_\varepsilon(x)| \leq \alpha(\varepsilon) e^{Lx}, \quad x \in [0, l]$$

baho hosil bo‘ladi. Bundan esa $Z_\varepsilon(x)$ ning nolga tekis yaqinlashishi kelib chiqadi. ■

2. Mexanika, amaliy matematika va nazariy fizikaning bir qator masalalari quyidagi

$$\varepsilon \frac{dy}{dx} = f(x, y), \quad (1.17.8)$$

$$y(x_0) = y_0 \quad (1.17.9)$$

kichik parametrli Koshi masalasiga keltiriladi. Bu masala $y = y(x, \varepsilon)$ yechimining $\varepsilon \rightarrow 0$ dagi limiti mavjudligini o‘rganish bilan shug‘ullanamiz.

Agar (1.17.8) differensial tenglamani

$$\frac{dy}{dx} = \frac{1}{\varepsilon} f(x, y) \quad (1.17.10)$$

ko‘rinishda ifodalasak, u holda bu tenglamaning o‘ng tomonidagi $F(x, y, \varepsilon) = \frac{1}{\varepsilon} f(x, y)$ funksiya $\varepsilon = 0$ da uzilishga ega. Shuning uchun bu yerda $y(x, \varepsilon)$ yechimning ε parametrغا nisbatan uzluksiz bog‘liqligi haqidagi teoremdan foydalanib bo‘lmaydi.

Bunday ko‘rinishdagi differensial tenglamalarni o‘rganishda $|\varepsilon|$ parametrning kichik qiymatlarida (1.17.8) tenglamada $\varepsilon y'$ hadni tashlab, $\varepsilon y' = f(x, y)$ differensial tenglamaning taqribiy yechimi sifatida $\varepsilon = 0$ dagi

$$f(x, y) = 0 \quad (1.17.11)$$

tenglamaning ildizini (yechimini) olish mumkinmi degan savolning tug‘ilishi tabiiy.

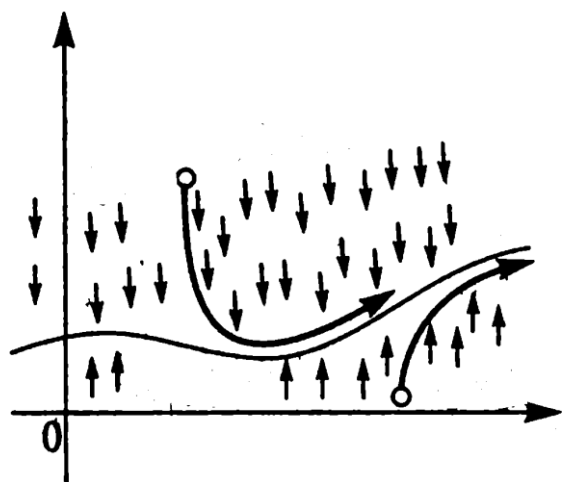
Aytalik (1.17.11) tenglama faqat bitta $y = \phi(x)$ ildizga ega bo‘lib, $\varepsilon > 0$ bo‘lsin. Ko‘rinib turibdiki $\varepsilon \rightarrow 0$ da (1.17.10) differensial tenglama yechimining $\frac{dy}{dx}$ hosilasi, $f(x, y) \neq 0$ shartni qanoatlantiruvchi har qanday

nuqtada absolyut qiymati bo‘yicha cheksiz ortadi. Bundan kelib chiqadiki, (1.17.10) tenglama integral chizig‘ining $f(x, y) \neq 0$ bo‘ladigan nuqtalariga o‘tkazilgan urinmalar, $\varepsilon \rightarrow 0$ da OY o‘q yo‘nalishiga parallel ravishda intiladi, ya’ni agar $f(x, y) > 0$ bo‘lsa, u holda (1.17.10) tenglamaning $y(x, \varepsilon)$

yechimi x o‘sishiga mos o‘sadi, chunki $\frac{dy}{dx} > 0$. Agar $f(x, y) < 0$ bo‘lsa, u

holda $y(x, \varepsilon)$ yechim x o‘sgan sari kamayadi, chunki $\frac{dy}{dx} < 0$ (1-chizma

qarang):

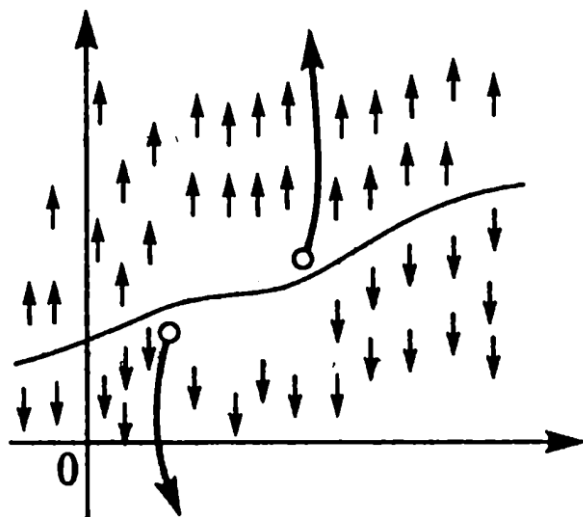


1-chizma

Quyidagi ikki holni qaraylik:

a) Aytaylik har bir tayinlangan x va y o'zgaruvchilarning ortishi natijasida $f(x, y)$ funksiya (1.17.11) tenglama $y = y(x)$ ildizining grafigidan o'tishda ishorasini "+" dan "-" ga o'zgartirsin. U holda (1.17.11) tenglamaning $y = y(x)$ ildizi turg'un bo'lib, (1.17.8) differensial tenglamaning $y(x, \varepsilon)$ yechimi $\varepsilon \rightarrow 0$ da $y = y(x)$ ga yaqinlashadi (chizma-1 ga qarang):

b) Aytaylik har bir tayinlangan x va y o'zgaruvchilarning ortishi natijasida $f(x, y)$ funksiya (1.17.11) tenglama $y = y(x)$ ildizining grafigidan o'tishda ishorasini - dan + ga o'zgartirsin. U holda (1.17.11) tenglamaning $y = y(x)$ ildizi noturg'un bo'lib, (1.17.8) differensial tenglamaning $y(x, \varepsilon)$ yechimini taqriban $y(x)$ bilan almashtirib bo'lmaydi (2-chizmaga qarang):



2-chizma

Yuqoridagi (1.17.11) tenglamaning $y = y(x)$ ildizi turg'un yoki noturg'unligini tekshirishda quyidagi yetarlilik shartlaridan foydalanish maqsadga muvofiqdir.

1. Agar (1.17.11) tenglama $y = y(x)$ ildizi ustida ushbu

$$\left. \frac{\partial f(x, y)}{\partial y} \right|_{y=y(x)} < 0$$

shart bajarilsa, u holda $y = y(x)$ ildiz turg'un bo'ladi.

2. Agar (1.17.11) tenglama $y = y(x)$ ildizi ustida quyidagi

$$\left. \frac{\partial f(x, y)}{\partial y} \right|_{y=y(x)} > 0$$

shart bajarilsa, u holda $y = y(x)$ ildiz noturg'un bo'ladi.

Shu jarayonlarni tavsiflovchi misollarni qaraylik.

1.17.1-misol. Ushbu

$$\varepsilon y' = x^2 - y, \quad y(x_0) = y_0, \quad x \in [x_0, \infty)$$

Koshi masalasining $y_\varepsilon(x) = y(x, \varepsilon)$, $\varepsilon > 0$ yechimini $\varepsilon \rightarrow +0$ da $y = x^2$ funksiyaga yaqinlashishini ko'rsating.

Yechish. Avvalo ushbu

$$f(x, y) = x^2 - y$$

funksiyani tuzib olamiz. Agar $\varepsilon = 0$ bo'lsa, u holda berilgan differensial tenglama quyidagi

$$f(x, y) = 0, \quad x^2 - y = 0$$

ko'rinishni oladi. Bu tenglamani yechib, uning $y = x^2$ ildizini topamiz. So'ngra quyidagi shartdan:

$$\left. \frac{\partial f(x, y)}{\partial y} \right|_{y=x^2} = \left[\frac{\partial}{\partial y} (x^2 - y) \right]_{y=x^2} = -1 < 0$$

kelib chiqadi, ya'ni $f(x, y) = 0$ tenglamaning $y = x^2$ ildizi turg'un bo'ladi. Shuning uchun

$$\lim_{\varepsilon \rightarrow +0} y_\varepsilon(x) = x^2, \quad x \in (x_0, \infty)$$

munosabat o'rinli bo'ladi.

Bu fikrga, berilgan Koshi masalasining yechimini to'g'ridan- to'g'ri topib ham kelish mumkin:

Haqiqatdan ham quyidagi

$$y_\varepsilon(x) := y(x, \varepsilon) = (y_0 - x_0^2 + 2\varepsilon x_0 - 2\varepsilon^2) e^{-\frac{x-x_0}{\varepsilon}} + x^2 - 2\varepsilon x + 2\varepsilon^2$$

funksiya $y(x_0) = y_0$ boshlang'ich shartni va berilgan differensial tenglamani qanoatlantiradi. Ko'rinib turibdiki, agar $x \in (x_0, \infty)$ bo'lsa u holda

$$\lim_{\varepsilon \rightarrow +0} e^{-\frac{x-x_0}{\varepsilon}} = 0, \quad x \in (x_0, \infty)$$

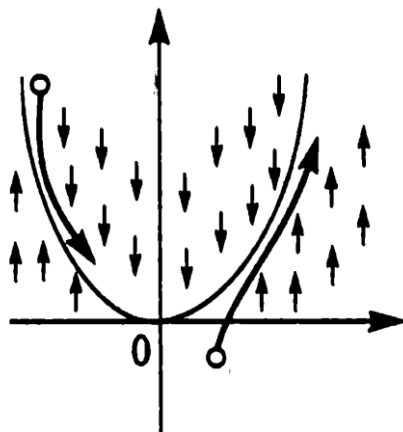
munosabat o'rinli bo'ladi. Shuning uchun ham $y_\varepsilon(x)$ yechimning ko'rinishidan

$$\lim_{\varepsilon \rightarrow +0} y_\varepsilon(x) = \lim_{\varepsilon \rightarrow +0} [(y_0 - x_0^2 + 2\varepsilon x_0 - 2\varepsilon^2) e^{-\frac{x-x_0}{\varepsilon}} + x^2 - 2x\varepsilon + 2\varepsilon^2] = x^2, \quad x \in (x_0, \infty)$$

kelib chiqadi. Ammo bu yaqinlashish tekis emas. Agar δ sonini $x_0 < \delta$ tengsizlikni qanoatlantiruvchi qilib tanlasak, u holda $\varepsilon \rightarrow +0$ da

$$e^{-\frac{x-x_0}{\varepsilon}} \Rightarrow 0, \quad x \in [\delta, \infty)$$

nolga tekis yaqinlashadi. Shuning uchun $\varepsilon \rightarrow +0$ da $y_\varepsilon(x) \Rightarrow x^2, x \in [\delta, \infty)$ tekis yaqinlashadi. Bu holda $[x_0, \delta]$ kesma chegaraviy qatlam vazifasini o'taydi (3-chizmaga qarang):



3-chizma

1.17.2-misol. Ushbu

$$\varepsilon y' = y - x, \quad \varepsilon > 0,$$

$$y(x_0) = y_0$$

Koshi masalasining $y_\varepsilon = y(x, \varepsilon)$ yehimini $\varepsilon \rightarrow +0$ da $y = x$ funksiyaga yaqinlashishini ko'rsating.

Yechish. Avvalo ushbu

$$f(x, y) = y - x$$

funksiyani tuzib olamiz. Agar $\varepsilon = 0$ bo'lsa, u holda berilgan tenglama

$$f(x, y) = 0, \quad y - x = 0$$

ko'rinishni oladi. Bu tenglamani yechib, uning $y = x$ ildizini topamiz. So'ngra quyidagi yetarlilik shartlarini tekshiramiz:

$$\frac{\partial f(x, y)}{\partial y} \Big|_{y=x} = \frac{\partial}{\partial y} [y - x] \Big|_{y=x} = +1 > 0.$$

Bu esa, $f(x, y) = 0$ tenglamaning $y = x$ ildizini noturg'un ekanligini ko'rsatadi. Shuning uchun $\varepsilon \rightarrow +0$ da $y_\varepsilon(x) \not\Rightarrow x, x \in (x_0, \infty)$ yaqinlashish bo'lmaydi. Bu fikrga berilgan Koshi masalasi yechimini topish orqali ham kelish mumkin. Ma'lumki, quyidagi

$$y_\varepsilon(x) = (y_0 - x_0 - \frac{1}{\varepsilon}) e^{-\frac{x-x_0}{\varepsilon}} + x + \frac{1}{\varepsilon}.$$

funksiya berilgan Koshi masalasining yechimidan iborat bo'ladi.

Ko'rinib turibdiki $\varepsilon \rightarrow +0$ da

$$e^{\frac{x-x_0}{\varepsilon}} \rightarrow \infty, \quad x \in (x_0, \infty)$$

munosabat o'rinli. Shuning uchun $\varepsilon \rightarrow +0$ da

$$y_\varepsilon(x) \not\rightarrow x, \quad x \in (x_0, \infty)$$

yaqinlashish bo'lmaydi.

2. Endi, ushbu

$$\begin{aligned} \varepsilon y' + ay &= f(x), \quad x \in [0, l] \\ y(0) &= y_0 \end{aligned} \quad (1.17.12)$$

ko'rinishdagi Koshi masalasini qaraylik. Bunda $\varepsilon > 0$, $a \neq 0$ va $f(x) \in [0, l]$ oraliqda berilgan uzluksiz funksiya.

Berilgan (1.17.12) Koshi masalasining yechimini $y_\varepsilon(x) = y(x, \varepsilon)$ orqali, $\varepsilon = 0$ bo'lgan holdagi yechimini esa $y_0(x) = y(x, 0)$ orqali belgilaylik. U holda $y_\varepsilon(x)$ funksiyaning $\varepsilon \rightarrow +0$ dagi limiti $y_0(x) = a^{-1}f(x)$ funksiya $x \in [0, l]$ oraliqda yaqinlashishi mumkinligini o'rganamiz.

1.17.3-teorema. Agar $a > 0$ bo'lsa, u holda har bir $x \in (0, l]$ larda

$$\lim_{\varepsilon \rightarrow +0} y_\varepsilon(x) = y_0(x), \quad y_0(x) = \frac{1}{a} f(x) \quad (1.17.13)$$

munosabat bajariladi.

Isbot. Berilgan (1.17.12) Koshi masalasining yechimini

$$y_\varepsilon(x) = y_0(x) + z_\varepsilon(x)$$

ko'rinishda izlaymiz. Bunda $z_\varepsilon(x)$ quyidagi

$$\begin{aligned} \varepsilon z'_\varepsilon(x) + az_\varepsilon(x) &= -\frac{\varepsilon}{a} f'(x), \\ z_\varepsilon(0) &= y_0 - \frac{1}{a} f(0) \end{aligned}$$

Koshi masalasining yechimidan iborat. Bu chiziqli tenglamani yechib

$$z_\varepsilon(x) = e^{-\frac{a}{\varepsilon}x} \left[y_0 - \frac{1}{a} f(0) \right] - \frac{1}{a} \int_0^x e^{\frac{a}{\varepsilon}(t-x)} f'(t) dt$$

formulani topamiz. U holda

$$y_\varepsilon(x) = y_0(x) + \left[y_0 - \frac{1}{a} f(0) \right] e^{-\frac{a}{\varepsilon}x} + O(\varepsilon), \quad \varepsilon \rightarrow +0 \quad (1.17.14)$$

baho o'rinli bo'ladi. Chunki $\varepsilon \rightarrow +0$ da

$$\begin{aligned} \left| \frac{1}{a} \int_0^x e^{\frac{a}{\varepsilon}(t-x)} f'(t) dt \right| &\leq \frac{1}{a} \max |f'(x)| \int_0^x e^{\frac{a}{\varepsilon}(t-x)} dt = \\ &= \frac{\varepsilon}{a^2} \max |f'(x)| \left(1 - e^{-\frac{a}{\varepsilon}x} \right) \leq \frac{\varepsilon}{a^2} \max |f'(x)| \end{aligned}$$

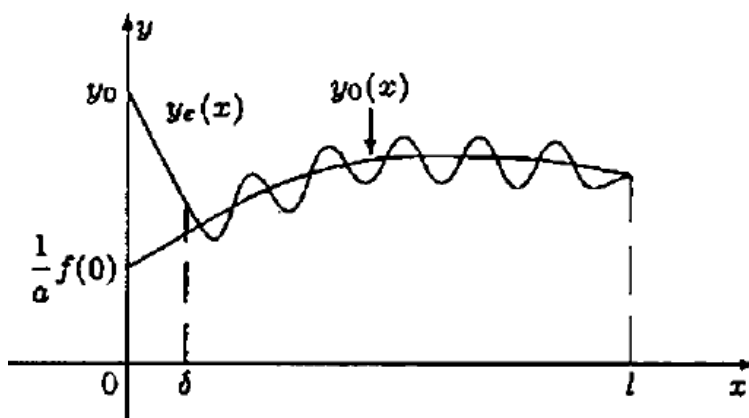
baho o‘rinli. Agar (1.17.14) asimptotikada $\varepsilon \rightarrow +0$ da ushbu

$$\lim_{\varepsilon \rightarrow +0} e^{-\frac{a}{\varepsilon}x} = 0, \quad \forall x \in (0, l], \quad a > 0$$

tenglikni inobatga olsak, undan (1.17.13) munosabat kelib chiqadi. Ammo $\varepsilon \rightarrow +0$ da $x \in (0, l]$ oraliqda $e^{-\frac{a}{\varepsilon}x}$ ($a > 0$) funksiya nolga tekis yaqinlashmaydi. Yuqoridagi (1.17.14) munosabatlardan ko‘rinadiki, agar $x \in [\delta, l]$, $0 < \delta < l$ (δ - ixtiyoriy tayinlangan son) bo‘lsa, u holda

$$y_\varepsilon(x) \Rightarrow y_0(x), \quad \varepsilon \rightarrow +0$$

tekis yaqinlashadi. Berilgan $[0, l]$ oraliqda $y_\varepsilon(x), \varepsilon \rightarrow +0$ da $y_0(x) = \frac{1}{a}f(x)$ funksiyaga nuqtali yaqinlashadi.



4-chizma

Agar $a < 0$ bo‘lsa, u holda $y_\varepsilon(x)$ yechim $\varepsilon \rightarrow +0$ da $y_0(x)$ funksiyaga yaqinlashmaydi.

Ushbu $[0, \delta]$, $0 < \delta < l$ kesmaga chegaraviy qatlam deyiladi.

Endi bu holatni atroflicha tekshiraylik. Buning uchun berilgan (1.17.12) differensial tenglamani ushbu

$$\varepsilon y' = F(x, y), \quad F(x, y) = f(x) - ay$$

ko‘rinishda yozib olamiz. U holda $y := y_0(x) = \frac{1}{a}f(x)$ funksiya $F(x, y) = 0$ tenglamaning ildizi bo‘ladi. Bu ildizni turg‘unlikka tekshiraylik:

$$\frac{\partial F(x, y)}{\partial y} = \frac{\partial}{\partial y}(f(x) - ay) = -a$$

agar $a > 0$ bo‘lsa, u holda ushbu

$$F'_y \Big|_{y=y_0(x)} = -a < 0$$

munosabat bajariladi. Shuning uchun $y = y_0(x)$ funksiya $F(x, y)$ tenglamaning turg‘un ildizi bo‘ladi. Bundan esa (1.17.13) munosabat kelib chiqadi.

Agar $a < 0$ bo‘lsa, u holda

$$F'_y|_{y=y_0(x)} = -a > 0$$

munosabat o‘rinli bo‘ladi. Demak $y = y_0(x)$ funksiya $F(x, y) = 0$ tenglamaning noturg‘un ildizi bo‘ladi. Shuning uchun $a < 0$ holda (1.17.13) munosabat bajarilmaydi.

Mustaqil yechish uchun mashqlar [21], §32, №960-967.

18-§. Hosilaga nisbatan yechilmagan sodda differensial tenglamalar

Biz ushbu

$$F(x, y, y') = 0 \quad (1.18.1)$$

ko‘rinishdagi differensial tenglamaning sodda ko‘rinishlarini integrallash bilan shug‘ullanamiz. Bu yerda $y = y(x)$ - noma’lum funksiya.

1. Aytaylik (1) differensial tenglamada F funksiya faqat y' ga bog‘liq, ya’ni

$$F(y') = 0 \quad (1.18.2)$$

bo‘lsin. Agar bu tenglama

$$y' = k_j, (j = 1, 2, \dots), \quad k_j = const \quad (1.18.3)$$

ko‘rinishdagi haqiqiy ildizlarga ega bo‘lsa, u holda (1.18.3) differensial tenglamadan

$$y = k_j x + c, \quad c = const,$$

yoki

$$k_j = \frac{y - c}{x},$$

kelib chiqadi. Bundan ko‘rinadiki (1.18.2) differensial tenglamaning umumiy integrali uchun quyidagi

$$F\left(\frac{y - c}{x}\right) = 0 \quad (1.18.4)$$

munosabat o‘rinli bo‘ladi.

2. Faraz qilaylik (1.18.1) differensial tenglamada F funksiya faqat x va y' o‘zgaruvchilarga bog‘liq, ya’ni

$$F(x, y') = 0 \quad (1.18.5)$$

ko‘rinishdagi differensial tenglama berilgan bo‘lsin.

1. Agar bu tenglamani y' ga nisbatan yechish mumkin bo‘lsa, u holda

$$y' = f_k(x), \quad k = 1, 2, \dots \quad (1.18.6)$$

ko‘rinishdagi differensial tenglamalar hosil bo‘ladi. Bu yerda $f_k(x)$ lar biror $[a, b]$ oraliqda aniqlangan va uzluksiz funksiyalar. Yuqoridagi (1.18.6) differensial tenglamani integrallab uning umumiy yechimlarini topamiz:

$$y = \int f_k(x) dx + c, \quad k = 1, 2, \dots; \quad c = const.$$

Bu yechimlar to'plamiga (1.18.5) differensial tenglamaning umumiy integrali deyiladi.

2. Faraz qilaylik (1.18.5) tenglama x o'zgaruvchiga nisbatan yechilgan, ya'ni

$$x = \phi(y') \quad (1.18.7)$$

ko'rinishdagi differensial tenglama berilgan bo'lsin. Bu holda (1.18.7) differensial tenglamani integrallash uchun quyidagi usuldan foydalanamiz. Shu maqsadda $y' = p$ deb belgilaymiz. Natijada (1.18.7) tenglama

$$x = \phi(p) \quad (1.18.8)$$

ko'rinishni oladi. Endi, belgilashdagi y' o'rniga uning $\frac{dy}{dx}$ qiymatini qo'yib

$$\frac{dy}{dx} = p, \quad dy = p dx$$

munosabatlarni topamiz. (1.18.8) tenglikni differensiallab

$$dx = \phi'(p) dp$$

munosabatni olamiz. Buni yuqoridagi tenglikka qo'ysak

$$dy = p dx = p \phi'(p) dp,$$

ya'ni

$$dy = p \phi'(p) dp$$

hosil bo'ladi. Oxirgi tenglikni integrallash natijasida

$$y = \int p \phi'(p) dp + c, \quad c = const \quad (1.18.9)$$

kelib chiqadi. Demak (1.18.5) tenglama

$$\begin{cases} x = \phi(p) \\ y = \int p \phi'(p) dp + c, \quad c = const \end{cases} \quad (1.18.10)$$

ko'rinishdagi yechimlar oilasiga ega bo'lar ekan.

3. Agar (1.18.1) differensial tenglama

$$F(y, y') = 0 \quad (1.18.11)$$

ko'rinishga ega bo'lsa, u holda yuqoridagi ikki hol takrorlanadi.

a) Aytaylik (1.18.11) tenglamani y' ga nisbatan yechish mumkin bo'lsa, u holda

$$y' = f_j(y), (j=1,2,\dots) \quad (1.18.12)$$

ko'rinishdagi differensial tenglamaga ega bo'lamiz. Bunda $f_k(y) \neq 0$ deb

$$\frac{dy}{f_k(y)} = dx, \quad x = \int \frac{dy}{f_k(y)} + c, \quad c = const \quad (1.18.13)$$

ko'rinishdagi yechimlarni topamiz. Agar $f_k(y) = 0$ tenglama $y = b_m$

ko'rinishdagi ildizga ega bo'lsa, u holda

$$y = b_m$$

uning yechimi bo'ladi.

b) Agar (1.18.11) tenglamadan $y = y(x)$ funksiyani

$$y = \phi(y') \quad (1.18.14)$$

topish mumkin bo'lsa, u holda $y' = p$ almashtirishdan foydalanish mumkin:

$$dy = y' dx, \quad dy = p dx, \quad dx = \frac{1}{p} dy,$$

$$\begin{cases} x = \int \frac{\phi'(p)}{p} dp + c, \\ y = \phi(p). \end{cases}$$

Bu esa (1.18.11) differensial tenglamaning umumiy yechimini beradi.

Mustaqil yechish uchun mashqlar [21], §8, №267-286.

19-§. Lagranj differensial tenglamasi

Ushbu

$$y = \phi(y')x + \psi(y') \quad (1.19.1)$$

ko'rinishdagi tenglamaga Lagranj differensial tenglamasi deyiladi.

Avvalo $\phi(y') \neq y'$ deb faraz qilamiz. Bu tenglamani integrallash uchun, $y' = p$ almashtirishdan foydalanamiz. U holda (1.19.1) differensial tenglama

$$y = \phi(p)x + \psi(p) \quad (1.19.2)$$

ko'rinishni oladi. Bu tenglikning ikki tomonini differensiallab

$$dy = d(\phi(p)x + \psi(p)) = d\phi(p)x + \phi(p)dx + d\psi(p) =$$

$$= \phi'(p)dp x + \phi(p)dx + \psi'(p)dp,$$

$$\frac{dy}{dx} = x\phi'(p)\frac{dp}{dx} + \phi(p) + \psi'(p)\frac{dp}{dx},$$

$$\frac{dy}{dx} = [x\phi'(p) + \psi'(p)]\frac{dp}{dx} + \phi(p),$$

$$p = [x\phi'(p) + \psi'(p)]\frac{dp}{dx} + \phi(p),$$

ya'ni

$$[x\phi'(p) + \psi'(p)]\frac{dp}{dx} = p - \phi(p)$$

o'zgaruvchilari ajraladigan differensial tenglamani hosil qilamiz. Oxirgi tenglikni quydagicha yozish mumkin:

$$\frac{dx}{dp} = \frac{\phi'(p)}{p - \phi(p)}x + \frac{\psi'(p)}{p - \phi(p)}. \quad (1.19.3)$$

Bu esa x o'zgaruvchiga nisbatan chiziqli bir jinslimas differensial tenglamadir. Uning umumiy integrali

$$\varphi(x, p, c) = 0, \quad c = const \quad (1.19.4)$$

ko'rinishga ega. Demak, Lagranj tenglamasining umumiy integrali

$$\begin{cases} y = \phi(p)x + \psi(p), \\ \varphi(x, p, c) = 0 \end{cases} \quad (1.19.5)$$

ko‘rinishda bo‘lar ekan. Bu sistemada p ni yo‘qotib Lagranj tenglamasining ushbu

$$\xi(x, y, c) = 0$$

ko‘rinishdagi umumiy integralini hosil qilamiz.

Agar $p - \phi(p) = 0$ bo‘lsa, u holda bu tenglamaning ildizlarini $p = p_i$ deb belgilasak, ushbu

$$y = \phi(p_i)x + \psi(p_i), \quad i = 1, 2, \dots$$

funksiyalar ham Lagranj tenglamasining yechimlaridan iborat bo‘ladi. Ammo bu yechimlar umumiy yechim formulasidan aniqlanmaydi.

20-§. Klero differensial tenglamasi

Ushbu

$$y = y'x + \psi(y') \quad (1.20.1)$$

ko‘rinishdagi tenglamaga Klero differensial tenglamasi deyiladi. Ko‘rinib turibdiki Klero tenglamasi Lagranj tenglamasining $\varphi(y') = y'$ bo‘lgandagi xususiy holdir. Klero tenglamasini yechimini topish jarayonida ham

$$y' = p$$

almashtirishdan foydalanamiz. U holda (1) tenglama

$$y = px + \psi(p). \quad (1.20.2)$$

ko‘rinishini oladi. Bu tenglikning ikki tomonini differensiallab

$$dy = d(px + \psi(p)) = xdp + pdx + d\psi(p) = xdp + pdx + \psi'(p)dp,$$

$$\frac{dy}{dx} = x \frac{dp}{dx} + p + \psi'(p) \frac{dp}{dx}, \quad p = x \frac{dp}{dx} + p + \psi'(p) \frac{dp}{dx}, \quad (x + \psi'(p)) \frac{dp}{dx} = 0$$

munosabatni hosil qilamiz. Bundan

$$\begin{cases} \frac{dp}{dx} = 0, \\ x + \psi'(p) = 0 \end{cases}$$

kelib chiqadi. Oxirgi munosabatdan

$$p = c, \quad c = \text{const} \quad \text{va} \quad x = -\psi'(p) \quad (1.20.3)$$

tengliklarni topamiz. Bundan esa

$$y = cx + \psi(c) \quad (1.20.4)$$

Klero tenglamasining umumiy yechimi kelib chiqadi.

Ushbu

$$\begin{cases} x = -\psi'(p), \\ y = -p\psi'(p) + \psi(p) \end{cases}$$

sistemada p ni yo‘qotib Klero tenglamasining quyidagi

$$\varphi(x, y) = 0$$

ko‘rinishdagi integralini hosil qilamiz.

Mustaqil yechish uchun mashqlar [21], §8, №287-296; [8], §8, №220-229.

21-§. Hosilaga nisbatan yechilmagan differensial tenglama uchun Koshi masalasi

Quyidagi misolga e’tibor qarataylik:

$$(y')^2 - 4x^2 = 0.$$

Bu tenglamani ushbu

$$(y' - 2x)(y' + 2x) = 0$$

ko‘rinishda yozib quyidagi

$$y' = 2x, \quad y' = -2x$$

differensial tenglamalarni hosil qilamiz. Ko‘rinib turibdiki

$$y(x) = x^2 + C_1, \quad y(x) = -x^2 + C_2$$

funksiyalar yuqoridagi differensial tenglamalarning umumiy yechimidan iborat bo‘ladi. Bu yerda C_1, C_2 - ixtiyoriy o‘zgarmas sonlar.

Endi ushbu

$$(y')^2 - 4x^2 = 0, \quad y(-1) = 1$$

Koshi masalasini qaraylik. Bu holda $y(x) = x^2 + C_1$ yechimning ko‘rinishidan va $y(-1) = 1$ boshlang‘ich shartdan $1 + C_1 = 1, C_1 = 0$, ya’ni $y = x^2$ - berilgan Koshi masalasining yechimini topamiz.

Ikkinchi tomondan, ya’ni $y(x) = -x^2 + C_2$ yechim formulasidan va $y(-1) = 1$ boshlang‘ich shartdan $-1 + C_2 = 1, C_2 = 2$, ya’ni $y = -x^2 + 2$ - berilgan Koshi masalasining yechimi kelib chiqadi. Bundan ko‘rinadiki berilgan Koshi masalasi ikkita, ya’ni $y = x^2, y = -x^2 + 2$ yechimlarga ega bo‘lar ekan.

Agar ushbu

$$(y')^2 - 4x^2 = 0, \quad y(-1) = 1, \quad y'(-1) = 2$$

ko‘rinishdagi Koshi masalasini qaraydigan bo‘lsak, u holda $y = -x^2 + 2$ funksiya uning yagona yechimidan iborat ekanligiga ishonch hosil qilamiz.

Qaralayotgan holda

$$F(x, y(x), y'(x)) = (y')^2 - 4x^2,$$

$$\frac{\partial F(x, y(x), y'(x))}{\partial y'} = 2y'.$$

Endi $(-1, y(-1), y'(-1))$ nuqtada F va $\frac{\partial F}{\partial y'}$ funksiyalarning qiymatini hisoblaymiz:

$$F(x, y(x), y'(x)) \Big|_{\substack{x=-1 \\ y(-1)=1 \\ y'(-1)=2}} = F(-1, 1, 2) = 4 - 4 = 0,$$

$$\frac{\partial F(x, y(x), y'(x))}{\partial y'} \Big|_{\substack{x=-1 \\ y(-1)=1 \\ y'(-1)=2}} = 2 \cdot 2 = 4 \neq 0.$$

Yuqoridagi mulohazalardan ko‘rinadiki, hosilaga nisbatan yechilmagan differensial tenglamalar uchun Koshi masalasi o‘zgacha qo‘yilar ekan.

Ushbu

$$F(x, y(x), y'(x)) = 0 \quad (1.21.1)$$

differensial tenglamani qaraylik. Bu yerda $F(x, y, y')$ uch o‘lchamli $G \subseteq R^3$ sohada berilgan uzluksiz funksiya.

1.21.1-ta’rif. Ushbu

$$F(x_0, y(x_0), y'(x_0)) = 0$$

shartni qanoatlantiruvchi $(x_0, y_0, y'_0) \in G$ nuqta berilgan bo‘lsin. (1.21.1) differensial tenglamaning

$$y(x_0) = y_0, \quad y'(x_0) = y'_0 \quad (1.21.2)$$

boshlang‘ich shartlarni qanoatlantiruvchi yechimini topishga Koshi masalasi deyiladi.

Quyidagi tasdiq (1.21.1), (1.21.2) Koshi masalasi yechimi mavjud va yagona bo‘lishligining yetarli shartlarini ifodalaydi.

1.21.1-teorema. Aytaylik, $F(x, y, y')$ funksiya $G \in R^3$ sohada uzluksiz differensiallanuvchi $F \in C^1(G)$ bo‘lib, $(x_0, y_0, y'_0) \in G$ nuqtada quyidagi

$$F(x_0, y_0, y'_0) = 0, \quad \frac{\partial F(x, y, y')}{\partial y'} \Big|_{\substack{x=x_0 \\ y(x_0)=y_0 \\ y'(x_0)=y'_0}} \neq 0, \quad (1.21.3)$$

shartlarni qanoatlantirsin. U holda shunday $h > 0$ soni topilib $x \in [x_0 - h, x_0 + h]$ oraliqda aniqlangan (1.21.1) differensial tenglamaning

(1.21.2) boshlang'ich shartni qanoatlantiruvchi yagona $y = y(x)$ yechimi mavjud bo'ladi.

Isbot. Quyidagi

$$\begin{cases} y'(x) = p, \\ F(x, y(x), p) = 0 \end{cases}$$

sistemani qaraylik. Berilgan $(x_0, y_0, y'_0) = (x_0, y_0, p_0) \in G$ nuqtada (1.21.3) shart, ya'ni

$$F(x_0, y_0, p_0) = 0, \quad \frac{\partial F(x_0, y_0, p_0)}{\partial p} \neq 0$$

o'rinli bo'lganligi uchun hamda $F \in C^1(G)$ funksiyaning uzluksiz differensiallanuvchi bo'lganligi sababli, shu nuqtaning biror $U(x_0, y_0, p_0) \subset G$ atrofida oshkormas funksiyaning mavjudligi va yagonaligi haqidagi teoremaning barcha shartlari bajariladi. Shuning uchun $U(x_0, p_0)$ va $U(p_0)$ atroflar topilib $\forall (x, y) \in U(x_0, p_0)$ lar uchun $F(x, y, p) = 0$ tenglamaning $p = f(x, y) \in U(p_0)$ ko'rinishdagi yagona yechimi mavjud. Bu $p = f(x, y)$ yechim $U(x_0, p_0)$ atrofda differensiallanuvchi bo'lib, $p_0 = f(x_0, y_0)$ shartni qanoatlantiradi. Ushbu $y' = f(x, y)$ tenglama biror $[x_0 - d_0, x_0 + d_0]$, $\exists d_0 > 0$ atrofda $y(x_0) = y_0$ boshlang'ich shartni qanoatlantiruvchi yagona yechimga ega bo'ladi, chunki

$$p = y'(x) = f(x, y(x))$$

bo'lib

$$F(x, y(x), y'(x)) \equiv 0, \quad y'(x_0) = f(x_0, y_0) = p_0 = y'_0.$$

22-§. Maxsus yechimlar va ularning mavjudligi

Differensial tenglama umumiy va xususiy yechimi tushunchasida aytiladiki, sohaning har bir berilgan nuqtasidan Koshi masalasining yagona yechimi o'tadi. Shunday qilib umumiy yoki xususiy yechimning har bir nuqtasida Koshi masalasi yagona yechimga ega bo'ladigan nuqta bo'ladi. Bunday nuqtaga oddiy nuqta deyiladi. Aks holda unga maxsus nuqta deb ataladi.

1.22.1-ta'rif. Differensial tenglama yechimining har bir nuqtasida Koshi masalasi yechimining yagonaligi buzilsa, bunday yechimga maxsus yechim deyiladi.

Bundan ko'rinadiki, maxsus yechim, differensial tenglamaning xususiy yechimi bo'la olmaydi va u umumiy yechim formulasi tarkibiga ham kirmaydi.

1.22.1-misol. Ushbu

$$\frac{dy}{dx} = 2\sqrt{y}$$

differensial tenglamani qaraylik. Bu tenglama $G = \{(x, y) \in R^2 : -\infty < x < \infty, 0 < y < \infty\}$ sohada ushbu

$$y(x) = (x + C)^2, \quad x \geq -C$$

ko‘rinishdagi umumiy yechimga ega. Berilgan differensial tenglama uchun $y(x) \equiv 0$ maxsus yechim bo‘ladi. Haqiqatan ham, ixtiyoriy $M(x_0, 0) \in R$ nuqtadan berilgan differensial tenglamaning kamida ikkita

$$y_1(x) \equiv 0, \quad y_2(x) = \begin{cases} 0, & x \leq x_0 \\ (x - x_0)^2, & x > x_0 \end{cases}$$

yechimi o‘tadi.

Aytaylik ushbu

$$\frac{dy}{dx} = f(x, y) \quad (1.22.1)$$

differensial tenglama quyidagi

$$\Phi(x, y, C) = 0 \quad (1.22.2)$$

ko‘rinishdagi integral chiziqlar oilasiga ega bo‘lsin. Bunda C - parametr.

1.22.2-ta’rif. Agar $l : (y = \varphi(x))$ chiziqning har bir nuqtasidagi urinma (1.22.2) ga tegishli kamida bitta chiziqning urinmasi bilan bir xil bo‘lsa, unga (1.22.2) chiziqlar oilasining o‘ramasi deb ataladi.

1.22.1-teorema. (1.22.2) ko‘rinishidagi integral chiziqlar oilasining o‘ramasi (1.22.1) differensial tenglamaning maxsus yechimi bo‘ladi.

1.22.2-misol. Ushbu

$$\frac{dy}{dx} = 2\sqrt{y}$$

differensial tenglamaning integral chiziqlari oilasi

$$y(x) = (x + C)^2, \quad x \geq -C$$

ko‘rinishda bo‘lishi ravshan. Bu yechimlar oilasining o‘ramasi $y(x) = 0$, ya’ni OX - absissa o‘qidan iborat. Shuning uchun $y(x) \equiv 0$ funksiya berilgan differensial tenglamaning maxsus yechimidan iborat bo‘ladi.

Endi hosilaga nisbatan yechilmagan

$$F(x, y(x), y'(x)) = 0 \quad (1.22.3)$$

differensial tenglamaning maxsus yechimlarini aniqlash bilan shug‘ullanamiz.

Uch o‘lchamli fazoning $(x, y, y') \in G \subset R^3$ sohasidagi har bir nuqtasida $F(x, y, y')$ funksiya 21-paragrafdagi teoremaning shartlarini qanoatlantirsa, u holda XOY tekislikning (x_0, y_0) nuqtasidan (1.22.3) tenglamaning bitta integral chizig‘i o‘tadi. Shuning uchun maxsus yechimni yagonalik buziladigan nuqtada izlash lozim.

Aytaylik, $F(x, y, y') \in C^1(G)$ bo'lib, (x_0, y_0) nuqtada yagonalik buzilsin. U holda biror y'_0 uchun quyidagi

$$F(x_0, y_0, y'_0) = 0, \quad \frac{\partial F(x_0, y_0, y'_0)}{\partial y'_0} = 0 \quad (1.22.4)$$

tengliklar bajariladi. Bunda y'_0 - oldindan berilmaganligi sababli, (x_0, y_0) nuqtani topish uchun (1.22.4) sistemadan y'_0 ni yo'qotamiz. Natijada $\varphi(x_0, y_0) = 0$ tenglama hosil bo'ladi.

1.22.3-ta'rif. (1.22.4) tenglamalar sistemasidan y'_0 ni yo'qotish natijasida hosil bo'lgan nuqtalarning geometrik o'rni, ya'ni $\varphi(x_0, y_0) = 0$ ga (1.22.3) differensial tenglamaning diskriminant chizig'i deyiladi.

Diskriminant chiziq yagonalik buziladigan nuqtalarning barchasini o'zida saqlaydi. Shu bilan bir qatorda, u boshqa nuqtalarni ham o'zida saqlashi mumkin.

Shunday qilib (1.22.3) differensial tenglamaning maxsus yechimlarini topish uchun;

1. Diskriminant chiziqlarni topish kerak;
2. Diskriminant chiziqlarni (1.22.3) tenglamaning yechimi ekanligini tekshirib ko'rish kerak;
3. Bu integral chiziqlarning har bir nuqtasida yagonalik buzilishini aniqlash, ya'ni tekshirilayotgan integral chiziqni maxsus yechim ta'rifini qanoatlantirishini tekshirish kerak.

(1.22.3) differensial tenglamaning yuqoridagi uchta shartni qanoatlantiruvchi yechimi, uning maxsus yechimi bo'ladi.

Maxsus yechimlarni izlashda ikki $y = \varphi(x)$ va $y = \psi(x)$ chiziqning urinish shartidan, ya'ni

$$\varphi(x) = \psi(x), \quad \varphi'(x) = \psi'(x)$$

tengliklardan foydalanishga to'g'ri keladi.

1.22.2-teorema. $y = \psi(x)$ chiziq (1.22.3) differensial tenglamaning maxsus yechimi bo'lishi uchun u (1.22.3) tenglama $\Phi(x, y, C) = 0$ yechimlari oilasining o'ramasi bo'lishi zarur va yetarli.

Yuqoridagi mulohazalardan ko'rinadiki, maxsus yechimni topish jarayonida diskriminant chiziqlarni berilgan differensial tenglamaning yechimi bo'lishini tekshirish talab qilinadi. Quyidagi tasdiqdan foydalansak bunday tekshirishga xojat qolmaydi.

1.22.3-teorema. Aytaylik $F(x, y, y')$ funksiya

- 1) $F(x, y, y') \in C^1(G)$
- 2) $\frac{\partial F(x, y, y')}{\partial x} \neq 0$

shartlarni qanoatlantirsin. U holda diskriminant chiziq (1.22.3) differensial tenglamaning maxsus yechimi bo‘lishi uchun

$$\begin{aligned} F(x, y, y') &= 0, \\ \frac{\partial F(x, y, y')}{\partial y'} &= 0, \\ \frac{\partial F(x, y, y')}{\partial y'} + y' \frac{\partial F(x, y, y')}{\partial y'} &= 0 \end{aligned}$$

tengliklarning bajarilishi zarur va yetarli.

1.22.1-misol. Ushbu

$$(y')^2 - 4y^3(1-y) = 0$$

differensial tenglamaning diskriminant chizig‘ini toping.

Yechish. Avvalo quyidagi

$$\begin{aligned} F(x, y, y') &= (y')^2 - 4y^3(1-y) = 0 \\ \frac{\partial F(x, y, y')}{\partial y'} &= 2y' = 0 \end{aligned}$$

tenglamalarni tuzib olamiz. Bu tenglamalarning ikkinchisidan $y' = 0$ ni topamiz. Buni birinchi tenglamaga qo‘yib

$$4y^3(1-y) = 0$$

diskriminant chiziqni hosil qilamiz. Oxirgi tenglamadan diskriminant chiziqning ikki shohchasini aniqlaymiz:

$$y = 0 \text{ va } y = 1.$$

Qaralayotgan holda bu topilgan $y = 0$ va $y = 1$ diskriminant chiziqlarning ikkalasi ham berilgan differensial tenglamaning yechimlari bo‘ladi.

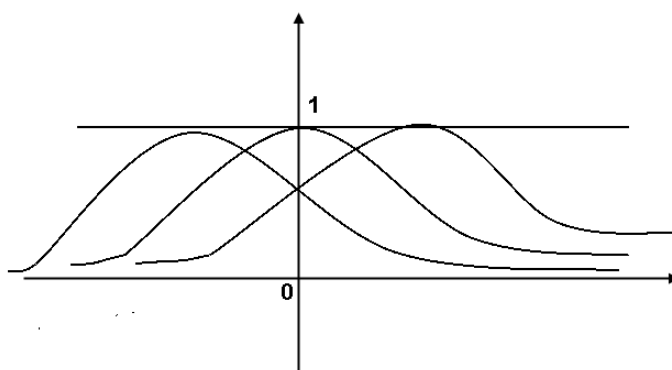
Qaysi chiziqda yagonalik buzilishini tekshirish maqsadida berilgan differensial tenglamaning boshqa yechimlarini ham topamiz. Buning uchun berilgan tenglamani y' ga nisbatan yechib

$$y' = \pm 2y\sqrt{y(1-y)}$$

differensial tenglamani hosil qilamiz. Bu tenglamani o‘zgaruvchilarini ajratish usulidan foydalanib yechsak, ushbu

$$y(x) = \begin{cases} y = 0, \\ y = 1, \\ y = [(x + C)^2 + 1]^{-1} \end{cases}$$

ko‘rinishdagi yechim hosil bo‘ladi. Ko‘rinib turibdiki, $y = 0$ chiziqda yagonalik buzilmaydi. Ammo $y = 1$ chiziqda esa yagonalik buziladi (5-chizmaga qarang).



5-chizma

Demak, $y(x)=1$ maxsus yechim bo‘ladi.

1.22.2-misol. Ushbu

$$(y-1)(y')^2 + 2y' - 1 = 0$$

differensial tenglamaning diskriminant chizig‘i topilsin.

Yechish. Avvalo quyidagi

$$F(x, y, y') \equiv (y-1)(y')^2 + 2y' - 1 = 0,$$

$$\frac{\partial F(x, y, y')}{\partial y'} \equiv 2(y-1)y' + 2 = 0$$

tenglamalarni hosil qilamiz. Bu sistemadan y' ni yo‘qotib $y=0$ diskriminant chiziqni topamiz. Bu esa berilgan differensial tenglamani qanoatlantirmaydi. Endi berilgan differensial tenglamaning boshqa yechimlarini topamiz. Buning uchun berilgan tenglamadan y' ni, topib ushbu

$$y' = \frac{1}{1 \pm \sqrt{y}}$$

differensial tenglamani hosil qilamiz. Bundan esa

$$y \pm \frac{2}{3} y^{\frac{2}{3}} = x + 8$$

kelib chiqadi. Ko‘rinib turibdiki $y=0$ diskriminant chiziq integral chiziqlar grafiklarining “uchli” nuqtalarining geometrik o‘rnidan iborat.

1.22.3-misol. Ushbu

$$y = x + 2y' - (y')^2$$

differensial tenglamaning maxsus yechimini toping.

Yechish. Berilgan differensial tenglama $y' = p$ almashtirish natijasida ushbu

$$y = x + 2p - p^2$$

ko‘rinishga keladi. Qaralayotgan holda, diskriminant chiziqni topish uchun quyidagi

$$F(x, y, y') = y - x - 2p + p^2 = 0,$$

$$\frac{\partial F(x, y, y')}{\partial y'} = -2 + 2p = 0$$

sistemani tuzib olamiz. Bu sistemada p parametrni yo‘qotamiz:

$$-2 + 2p = 0, \quad 2p = 2, \quad p = 1,$$

$$y - x - 2 \cdot 1 + (1)^2 = 0,$$

$$y - x - 2 + 1 = 0,$$

$$y = x + 1.$$

Bu esa biz izlagan diskriminat chiziq bo‘ladi.

Endi $y = x + 1$ chiziqning maxsus yechim bo‘lishini tekshirib ko‘ramiz:

1. $y = x + 1$ -diskriminant chiziq, berilgan differensial tenglamani qanoatlantirishini tekshiramiz:

$$x + 2y' - (y')^2 = x + 2 \cdot 1 - (1)^2 = x + 1, \text{ ya'ni } x + 1 \equiv x + 1.$$

2. Bu diskriminant chiziq, ya'ni $y = x + 1$ ning nuqtalarida yagonalik buzilishini tekshiramiz:

$$\begin{cases} \frac{dy}{dx} = p, & dy = p dx, \\ y = x + 2p - p^2, \end{cases}$$

$$\begin{cases} dy = dx + 2dp - 2p dp, \\ dy = p dx, \end{cases}$$

$$p dx = dx + 2dp - 2p dp,$$

$$(p - 1)dx - 2dp + 2p dp = 0,$$

$$(p - 1)dx + 2(p - 1)dp = 0,$$

$$(p - 1)[dx + 2dp] = 0.$$

Bundan

$$p = 1, \quad dx = -2dp$$

kelib chiqadi. Oxirgi differensial tenglamani integrallab

$$x = -2p + C$$

topamiz. Demak berilgan differensial tenglama

$$y = x + 1 \quad \text{va} \quad \begin{cases} x = -2p + C, \\ y = x + 2p - p^2 \end{cases}$$

ko‘rinishdagi yechimlarga ega bo‘lar ekan. Oxirgi sistemadan p parametrni yo‘qotamiz:

$$p = \frac{C - x}{2},$$

$$y = x + C - x - \frac{(C - x)^2}{4} = C - \frac{(C - x)^2}{4}.$$

Shunday qilib ushbu

$$y(x) = y_1 = x + 1, \quad y(x) = y_2 = C - \frac{(C - x)^2}{4}$$

funksiyalar berilgan differensial tenglamaning yechimlari bo‘lar ekan. Bu yechimlar grafiklarining bir-biriga urinishini tekshiramiz. Buning uchun $y_1(x_0) = y_2(x_0)$, $y_1'(x_0) = y_2'(x_0)$ tengliklardan foydalanamiz.

$$\begin{cases} x_0 + 1 = C - \frac{(C - x_0)^2}{4}, \\ 1 = \frac{C - x_0}{2}, \end{cases}$$

$$C = x_0 + 2,$$

$$x_0 + 1 = x_0 + 2 - \frac{(x_0 + 2 - x_0)^2}{4} = x_0 + 2 - 1 = x_0 + 1,$$

$$x_0 + 1 = x_0 + 1.$$

Oxirgi ayniyat barcha x_0 lar uchun o‘rinli. Demak, $y = x + 1$ chiziq, absissasi $x = x_0$ nuqtada

$$y = C - \frac{(C - x)^2}{4}$$

parabolalarning biri ($C = 2 + x_0$) bilan urinishga ega. Shuning uchun $y = x + 1$ diskriminant chiziqning barcha nuqtalarida yagonalik buzilganligi sababli, u maxsus yechim bo‘ladi.

Mustaqil yechish uchun mashqlar [21], §8, №241-266; [8], §11, №261-274.

II BOB. YUQORI TARTIBLI DIFFERENSIAL TENGLAMALAR

1-§. n-tartibli differensial tenglamalar

Ushbu

$$y^{(n)} = f(x, y, y', y'', \dots, y^{(n-1)}) \quad (2.1.1)$$

ko‘rinishdagi tenglamaga n-tartibli hosilaga nisbatan yechilgan oddiy differensial tenglamaning umumiy ko‘rinishi deyiladi.

Quyidagi belgilashlarni kiritaylik:

$$y(x) \equiv y^{(0)}, \quad y'(x) \equiv y^{(1)}, \dots, y^{(k)}(x) \equiv y^{(k)}, \dots, y^{(n-1)}(x) \equiv y^{(n-1)},$$

$$y = (y^{(0)}, y^{(1)}, \dots, y^{(n-1)})^T \in R^n, \quad (x, y) \in R^{n+1}.$$

Endi, R^{n+1} $(n+1)$ -o‘lchamli fazoda quyidagi

$$G = \{(x, y) \in R^{n+1} : |x - x_0| \leq a, |y^{(k)} - y_0^{(k)}| \leq b, k = \overline{0, n-1}\}$$

sohani olaylik. Bu yerda a, b —o‘zgarmas sonlar bo‘lib, $y_0^{(k)} = y^{(k)}(x_0)$, $k = \overline{0, n-1}$.

2.1.1-ta’rif. Aytaylik, (2.1.1) ko‘rinishidagi oddiy differensial tenglama berilgan bo‘lib $f(x, y, y', y'', \dots, y^{(n-1)})$ funksiya $G \subset R^{n+1}$ sohada aniqlangan bo‘lsin. Agar $I = [x_0 - a, x_0 + a]$ oraliqda aniqlangan biror $y = \varphi(x)$ funksiya uchun quyidagi

1. $\varphi(x) \in C^{(n)}(I)$
2. $(x, \varphi(x), \varphi'(x), \varphi''(x), \dots, \varphi^{(n-1)}(x)) \in G, \forall x \in I$
3. $\varphi^{(n)}(x) = f(x, \varphi(x), \varphi'(x), \varphi''(x), \dots, \varphi^{(n-1)}(x)), \forall x \in I$

shartlar bajarilsa, $y = \varphi(x)$ funksiya $I = [x_0 - a, x_0 + a]$ oraliqda (2.1.1) differensial tenglamaning yechimi deyiladi.

2.1.2-ta’rif. (2.1.1) differensial tenglamaning ushbu

$$y(x_0) = y_0, \quad y'(x_0) = y'_0, \dots, y^{(n-1)}(x_0) = y_0^{(n-1)} \quad (2.1.2)$$

boshlang‘ich shartlarni qanoatlantiruvchi $y = y(x)$ yechimini topishga Koshi masalasi deyiladi. Bu yerda $(x_0, y_0, y'_0, y''_0, \dots, y_0^{(n-1)}) \in G$ berilgan nuqta.

2.1.1-teorema. Agar $f(x, y, y', y'', \dots, y^{(n-1)})$ funksiya $G \subset R^{n+1}$ sohada aniqlangan va uzluksiz bo‘lib, $y, y', y'', \dots, y^{(n-1)}$ o‘zgaruvchilar bo‘yicha

$$\left| f(x, y_1, y'_1, y''_1, \dots, y_1^{(n-1)}) - f(x, y_2, y'_2, y''_2, \dots, y_2^{(n-1)}) \right| \leq L \sum_{j=0}^{n-1} |y_1^{(j)} - y_2^{(j)}|, \quad \exists L > 0 \quad (2.1.3)$$

tengsizlikni qanoatlantirsa, u holda shunday $h > 0$ soni mavjud bo‘lib, (2.1.1)+(2.1.2) Koshi masalasining $I = [x_0 - h, x_0 + h]$ oraliqda aniqlangan yagona yechimi mavjud bo‘ladi.

Odatda bu yerdagi $L = \text{const} > 0$ - Lipshits o'zgarishi, (2.1.3) esa Lipshits sharti deyiladi.

Bu teoremaning isbotini qisuvchi akslantirishlar prinspidan foydalanib ko'rsatish mumkin. Uni biz bu yerda keltirmaymiz.

2-§. Ayrim n-tartibli differensial tenglamalarni yechish

Ushbu

$$y^{(n)} = f(x), \quad f(x) \in C[a, b] \quad (2.2.1)$$

differensial tenglamaning umumiy yechimini topish bilan shug'ullanamiz.

1. Ketma-ket integrallash usuli.

Avvalo (2.2.1) tenglamani

$$\left(y^{(n-1)}\right)' = f(x) \quad (2.2.2)$$

ko'rinishda yozib olamiz. Endi ixtiyoriy $\forall x_0 \in [a, b]$ nuqtani olib (2.2.2) differensial tenglamaning x_0 dan x gacha integrallab, ushbu

$$y^{(n-1)} = \int_{x_0}^x f(t) dt + c_1$$

tenglikni hosil qilamiz. Bu yerda C_1 ixtiyoriy haqiqiy son. Bu munosabatni yana integrallab ushbu

$$y^{(n-2)} = \int_{x_0}^x \int_{x_0}^x f(x) dx dx + c_1 x + c_2$$

tasvirni topamiz. Yuqoridagi jarayonni davom qildirib

$$y(x) = \underbrace{\int_{x_0}^x \int_{x_0}^x \dots \int_{x_0}^x}_{n \text{ marta}} f(x) \underbrace{dx dx \dots dx}_{n \text{ marta}} + c_1 \frac{x^{n-1}}{(n-1)!} + c_2 \frac{x^{n-2}}{(n-2)!} + \dots + c_{n-1} x + c_n \quad (2.2.3)$$

(2.2.1) differensial tenglamaning umumiy yechimini topishga muvaffaq bo'lamiz.

Quyidagi

$$y(x) = \underbrace{\int_{x_0}^x \int_{x_0}^x \dots \int_{x_0}^x}_{n \text{ marta}} f(x) \underbrace{dx dx \dots dx}_{n \text{ marta}} = \frac{1}{(n-1)!} \int_{x_0}^x f(t) (x-t)^{n-1} dt$$

formuladan foydalanib (2.2.3) munosabatni

$$y(x) = \frac{1}{(n-1)!} \int_{x_0}^x f(t)(x-t)^{n-1} dt + c_1 \frac{x^{n-1}}{(n-1)!} +$$

$$+ c_2 \frac{x^{n-2}}{(n-2)!} + \dots + c_{n-1}x + c_n$$
(2.2.4)

ko‘rinishda yozish mumkin. Bu yerda $C_j, j = \overline{1, n}$ - ixtiyoriy haqiqiy sonlar.

2. **Koshi usuli.** Avvalo ixtiyoriy $\forall t \in (a, b)$ nuqtani tanlab olamiz va quyidagi Koshi masalasini qaraymiz:

$$y^{(n)} = 0, \tag{2.2.5}$$

$$y(t) = 0, y'(t) = 0, \dots, y^{(n-2)}(t) = 0, y^{(n-1)}(t) = 1 \tag{2.2.5'}$$

So‘ngra, ushbu

$$y^{(n)} = 0$$

differensial tenglamaning umumiy yechimini ketma-ket integrallash natijasida topamiz:

$$y^{(n-1)}(x) = c_1,$$

$$y^{(n-2)}(x) = c_1x + c_2,$$

$$y^{(n-3)}(x) = c_1 \frac{x^2}{2!} + c_2x + c_3, \tag{2.2.6}$$

.....

$$y(x) = c_1 \frac{x^{n-1}}{(n-1)!} + c_2 \frac{x^{n-2}}{(n-2)!} + \dots + c_{n-1}x + c_n.$$

Bu yerdagi c_1, c_2, \dots, c_n o‘zgarmlarining qiymatlarini boshlang‘ich shartlardan foydalanib topish mumkin:

$$c_1 \frac{t^{n-1}}{(n-1)!} + c_2 \frac{t^{n-2}}{(n-2)!} + \dots + c_{n-1}t + c_n = 0,$$

$$c_1 \frac{t^{n-2}}{(n-2)!} + c_2 \frac{t^{n-3}}{(n-3)!} + \dots + c_{n-2}t + c_{n-1} = 0, \tag{2.2.7}$$

.....

$$c_1 \frac{t^3}{3!} + c_2 \frac{t^2}{2!} + c_3 \frac{1}{1!}t + c_4 = 0,$$

$$c_1 \frac{t^2}{2!} + c_2 \frac{t}{1!} + c_3 = 0,$$

$$c_1 \frac{t}{1!} + c_2 = 0,$$

$$c_1 = 1.$$

(2.2.7) sistemani pastdan yuqoriga qarab ketma-ket yechsak $c_j, j=1, n$ o'zgarmaslarning qiymatlari topiladi:

$$c_1 = 1, c_2 = -\frac{t}{1!}, c_3 = \frac{t^2}{2!}, c_4 = -\frac{t^3}{3!}, \dots, c_n = \frac{(-1)^{n-1} t^{n-1}}{(n-1)!}.$$

O'zgarmaslarning bu qiymatlarini (2.2.6) tenglikning o'ng tomoniga qo'yib, ushbu

$$y \equiv F(x, t) = \frac{x^{n-1}}{(n-1)!} - \frac{1}{(n-2)!} tx^{n-2} + \dots + \frac{(-1)^{n-1}}{(n-1)!} t^{n-1} = \frac{(x-t)^{n-1}}{(n-1)!}$$

Koshi funksiyasini topamiz.

2.2.1-lemma. Ushbu

$$\bar{y}(x) = \int_{x_0}^x F(x, t) f(t) dt = \int_{x_0}^x \frac{(x-t)^{n-1}}{(n-1)!} f(t) dt \quad (2.2.8)$$

funksiya (2.2.1) differensial tenglamaning quyidagi

$$\bar{y}(x_0) = 0, \bar{y}'(x_0) = 0, \dots, \bar{y}^{(n-1)}(x_0) = 0 \quad (2.2.8')$$

boshlang'ich shartlarni qanoatlantiruvchi xususiy yechimdan iborat bo'ladi.

Bu lemmani isbotlashni o'quvchiga havola qilamiz.

Endi, (2.2.1) differensial tenglama bir jinsli

$$y^{(n)} = 0$$

qismining umumiy yechimi

$$\bar{\bar{y}}(x) = A_0 + A_1 x + A_2 x^2 + \dots + A_{n-1} x^{n-1}$$

ko'rinishda bo'lishini inobatga olsak, u holda (2.2.1) differensial tenglamaning umumiy yechimi

$$y(x) = \bar{y}(x) + \bar{\bar{y}}(x) = A_0 + A_1 x + A_2 x^2 + \dots + A_{n-1} x^{n-1} + \int_{x_0}^x \frac{(x-t)^{n-1}}{(n-1)!} f(t) dt \quad (2.2.9)$$

ko'rinishda bo'lishi kelib chiqadi.

2.2.1-misol. Ushbu

$$y'' = e^{-x^2}$$

differensial tenglamaning umumiy yechimini toping.

Yechish. Qaralayotgan differensial tenglamaning umumiy yechimini (2.2.9) formuladan foydalanib topish mumkin:

$$y(x) = \int_{x_0}^x (x-t) e^{-t^2} dt + A_0 x + A_1$$

Mustaqil yechish uchun mashqlar [8], §14, №327-363.

3-§. n-tartibli chiziqli differensial tenglamalar

n-tartibli oddiy differensial tenglamalarning muhim xususiy hollaridan biri, n-tartibli chiziqli differensial tenglama bo'lib, u quyidagi ko'rinishda yoziladi:

$$y^{(n)} + p_1(x)y^{(n-1)} + \dots + p_{n-1}(x)y' + p_n(x)y = g(x). \quad (2.3.1)$$

Bunga n-tartibli bir jinsli bo'lmagan differensial tenglama deyiladi.

Agar (2.3.1) tenglamada $g(x) = 0$, ya'ni

$$y^{(n)} + p_1(x)y^{(n-1)} + \dots + p_{n-1}(x)y' + p_n(x)y = 0 \quad (2.3.2)$$

bo'lsa, bunga n-tartibli chiziqli bir jinsli differensial tenglama deyiladi. Bu yerda $p_j(x), g(x) \in C[a, b]$, $j = \overline{1, n}$ -berilgan uzluksiz funksiyalarga mos ravishda (2.3.1) tenglamaning koeffitsiyentlari va uning o'ng tomoni deyiladi.

2.3.1-ta'rif. (2.3.1) differensial tenglamani ayniyatga aylantiruvchi $y = \varphi(x) \in C^n[a, b]$ funksiyaga uning yechimi deyiladi.

2.3.1-lemma. Agar $g(x) = g_1(x) + g_2(x)$ ko'rinishda bo'lib, $y_1(x)$ va $y_2(x)$ funksiyalar mos ravishda ushbu

$$y_1^{(n)} + p_1(x)y_1^{(n-1)} + \dots + p_{n-1}(x)y_1' + p_n(x)y_1 = g_1(x)$$

$$y_2^{(n)} + p_1(x)y_2^{(n-1)} + \dots + p_{n-1}(x)y_2' + p_n(x)y_2 = g_2(x)$$

differensial tenglamaning yechimidan iborat bo'lsa, u holda $y(x) = y_1(x) + y_2(x)$ funksiya (2.3.1) tenglamaning yechimi bo'ladi.

2.3.1-natija. Agar $y_1(x), y_2(x)$ funksiyalar (2.3.2) bir jinsli tenglamaning yechimlari bo'lib, C_1, C_2 -ixtiyoriy o'zgarmas sonlar bo'lsa, u holda $y = c_1y_1(x) + c_2y_2(x)$ funksiya (2.3.2) tenglamaning yechimi bo'ladi.

Bu ikki tasdiqqa (2.3.1) tenglama uchun superpozitsiya prinsipi deyiladi. Superpozitsiya prinsipi faqat chiziqli differensial tenglamaga xos xususiyatdir.

Endi (2.3.1) differensial tenglamaga qo'yilgan

$$y(x_0) = y_0^{(0)}, y'(x_0) = y_0^{(1)}, \dots, y^{(n-1)}(x_0) = y_0^{(n-1)}, \forall x_0 \in [a, b] \quad (2.3.3)$$

Koshi masalasini qaraymiz. Bunda $x_0 \in [a, b]$ va $y_0^{(0)}, y_0^{(1)}, y_0^{(2)}, \dots, y_0^{(n-1)}$ berilgan sonlar.

2.3.1-teorema. Faraz qilaylik, $p_j(x), j = \overline{1, n}$ va $g(x), x \in [a, b]$ funksiyalar uzluksiz bo'lib, $x_0 \in [a, b]$ bo'lsin. U holda $y_0^{(j)}, j = \overline{0, n-1}$ -berilgan sonlarning ixtiyoriy qiymatlarida (2.3.1), (2.3.3) Koshi masalasining $[a, b]$ kesmada aniqlangan yagona yechimi mavjud.

Isbot. Avvalo (2.3.1) differensial tenglamani ushbu

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)}) \quad (2.3.4)$$

ko‘rinishda yozib olamiz. Bu yerda

$$f(x, y, y', \dots, y^{(n-1)}) = g(x) - p_1(x)y^{(n-1)} - p_2(x)y^{(n-2)} - \dots - p_n(x)y. \quad (2.3.4')$$

Bu $f(x, y, y', \dots, y^{(n-1)})$ funksiya 2.1.1-teoremaning shartlarini qanoatlantirishini ko‘rsatamiz. Aniqlanishiga ko‘ra, bu $f(x, y, y', \dots, y^{(n-1)})$ funksiya ushbu

$$G = \left\{ (x, y, y', \dots, y^{(n-1)}) \in R^{n+1} : x \in [a, b], -\infty < y^{(j)} < \infty, j = \overline{0, n-1} \right\}$$

sohada aniqlangan va uzluksiz bo‘lib $y^{(j)}, j = \overline{0, n-1}$ o‘zgaruvchilar bo‘yicha Lipshits shartini qanoatlantiradi. Haqiqatan ham, quyidagi

$$\frac{\partial f}{\partial y^{(j)}} = -p_{n-j}(x), \quad j = \overline{0, n-1}$$

munosabatdan

$$\max_{x \in [a, b]} \left| \frac{\partial f}{\partial y^{(j)}} \right| = L_j, \quad j = \overline{0, n-1}$$

kelib chiqadi. Chunki $p_{n-j}(x), j = \overline{0, n-1}$ funksiyalar $[a, b]$ kesmada uzluksiz. Endi, ushbu

$$L = \max(L_0, L_1, \dots, L_{n-1})$$

belgilashni olsak, u holda (2.3.4') tenglik orqali aniqlangan $f(x, y, y', \dots, y^{(n-1)})$ funksiya $L_1 > 0$ o‘zgarimas bilan $(y, y', \dots, y^{(n-1)})$ o‘zgaruvchilar bo‘yicha Lipshits shartini qanoatlantirishiga ishonch hosil qilamiz. Shuning uchun (2.3.1), (2.3.3) Koshi masalasining $[a, b]$ kesmada aniqlangan yechimi mavjud va yagona bo‘ladi. ■

2.3.1-natija. Ushbu

$$y^{(n)} + p_1(x)y^{(n-1)} + \dots + p_{n-1}(x)y' + p_n(x)y = 0, \quad p_j(x) \in C[a, b], \quad j = \overline{1, n} \quad (2.3.5)$$

$$y(x_0) = y'(x_0) = \dots = y^{(n-1)}(x_0) = 0, \quad \forall x_0 \in [a, b]$$

Koshi masalasi yagona $y(x) \equiv 0, \forall x_0 \in [a, b]$ yechimga ega. ■

Isbot. Ko‘rinib turibdiki $y(x) \equiv 0$ funksiya (2.3.5) Koshi masalasining yechimidan iborat. Yechimning yagonaligidan 2.3.1-natijaning isboti kelib chiqadi. ■

Quyidagi

$$L[y] = y^{(n)} + p_1(x)y^{(n-1)} + \dots + p_{n-1}(x)y' + p_n(x)y \quad (2.3.6)$$

belgilash natijasida (2.3.1) va (2.3.2) differensial tenglamalarni ushbu

$$L[y] = g(x), \quad (2.3.7)$$

$$L[y] = 0 \quad (2.3.8)$$

ko‘rinishda yozish mumkin. Bu yerda $L[y]$ ifodaga differensial operator deyiladi. Endi differensial operatorning ayrim xossalari bilan tanishamiz.

2.3.2-lemma. O‘zgarimas ko‘paytuvchini operator belgisidan tashqariga chiqarish mumkin, ya’ni

$$L[cy] = cL[y], \quad c = \text{const}.$$

Isbot.

$$\begin{aligned} L[cy] &= (cy)^{(n)} + p_1(x)(cy)^{(n-1)} + \dots + p_{n-1}(x)(cy)' + p_n(x)(cy) = \\ &= cy^{(n)} + cp_1(x)y^{(n-1)} + \dots + cp_{n-1}(x)y' + cp_n(x)y = \\ &= c(y^{(n)} + p_1(x)y^{(n-1)} + \dots + p_{n-1}(x)y' + p_n(x)y) = cL[y] \end{aligned}$$

2.3.3-lemma. Ushbu

$$L[y_1 + y_2] = L[y_1] + L[y_2]$$

tenglik o'rinli.

Isbot.

$$\begin{aligned} L[y_1 + y_2] &= (y_1 + y_2)^{(n)} + p_1(x)(y_1 + y_2)^{(n-1)} + \dots \\ &+ p_{n-1}(x)(y_1 + y_2)' + p_n(x)(y_1 + y_2) = y_1^{(n)} + p_1(x)y_1^{(n-1)} + \dots + p_{n-1}(x)y_1' + \\ &+ p_n(x)y_1 + y_2^{(n)} + p_1(x)y_2^{(n-1)} + \dots + p_{n-1}(x)y_2' + p_n(x)y_2 = L[y_1] + L[y_2]. \end{aligned}$$

2.3.2-natija. Quyidagi

$$L\left(\sum_{j=1}^m c_j y_j\right) = \sum_{j=1}^m c_j L[y_j]$$

tenglik o'rinli. Bu yerda $c_j = \text{const}$, $j = \overline{1, m}$.

2.3.2-teorema. Agar $y = y(x)$ funksiya $[a, b]$ kesmada (2.3.8) bir jinsli differensial tenglamaning yechimi bo'lsa, u holda $y_1 = cy(x)$, $c = \text{const}$ funksiya ham (2.3.8) tenglamaning yechimi bo'ladi.

Isbot. Teorema shartiga ko'ra $L[y] = 0$. Bundan $L[y_1] = L[cy(x)] = cL[y] = 0$ kelib chiqadi.

2.3.3-teorema. Agar $y_1 = y_1(x)$, $y_2 = y_2(x)$ funksiyalar $[a, b]$ kesmada (2.3.8) bir jinsli differensial tenglamaning yechimi bo'lsa, u holda

$$y = c_1 y_1(x) + c_2 y_2(x), \quad c_j = \text{const}, \quad j = 1, 2$$

funksiya ham $[a, b]$ kesmada (2.3.8) tenglamaning yechimi bo'ladi.

Isbot. Teorema shartiga ko'ra $L[y_1] = 0$, $L[y_2] = 0$.

$$L[y] = L[c_1 y_1 + c_2 y_2] = c_1 L[y_1] + c_2 L[y_2] = 0$$

2.3.3-natija. Agar $y_1(x), y_2(x), \dots, y_n(x)$ funksiyalar $[a, b]$ kesmada (2.3.8) bir jinsli differensial tenglamaning yechimlaridan iborat bo'lsa, u holda ushbu

$$y = \sum_{j=1}^n c_j y_j(x), \quad c_j = \text{const}, \quad j = \overline{1, n}$$

funksiya ham (2.3.8) tenglamaning yechimi bo'ladi.

4-§. Vronskiy determinanti

2.4.1-ta'rif. Agar quyidagi shartni qanoatlantiruvchi

$$\exists a_1, a_2, \dots, a_n \in R, \quad \sum_{j=1}^n a_j^2 \neq 0$$

sonlar topilib, ushbu

$$a_1\varphi_1(x) + a_2\varphi_2(x) + \dots + a_n\varphi_n(x) = 0, \quad \forall x \in [a, b] \quad (2.4.1)$$

munosabat bajarilsa, $\varphi_1(x), \varphi_2(x), \dots, \varphi_n(x)$ funksiyalarga $[a, b] \subset R$ oraliqda chiziqli bog'langan funksiyalar deyiladi.

Agar (2.4.1) tenglik a_1, a_2, \dots, a_n -o'zgarmlarning faqat nolga teng qiymatida bajarilsa $\varphi_1(x), \varphi_2(x), \dots, \varphi_n(x)$ funksiyalarga chiziqli bog'lanmagan funksiyalar deyiladi.

2.4.1-teorema. Agar $\varphi_1(x), \varphi_2(x), \dots, \varphi_n(x)$ funksiyalar (a, b) intervalda chiziqli bog'langan bo'lsa, u holda ularning ichidan bittasi qolganlarining chiziqli kombinatsiyasi orqali ifodalanadi.

Isbot. Aytaylik, ushbu

$$a_1\varphi_1(x) + a_2\varphi_2(x) + \dots + a_n\varphi_n(x) = 0, \quad \forall x \in [a, b]$$

tenglik $a_n \neq 0$ bo'lganda bajarilsin. U holda

$$\varphi_n(x) = -\frac{a_1}{a_n}\varphi_1(x) - \frac{a_2}{a_n}\varphi_2(x) - \dots - \frac{a_{n-1}}{a_n}\varphi_{n-1}(x)$$

munosabatga ega bo'lamiz. Bu esa $\varphi_n(x)$ funksiya $\varphi_1(x), \varphi_2(x), \dots, \varphi_{n-1}(x)$ funksiyalarning chiziqli kombinatsiyalaridan iborat ekanligini ko'rsatadi.

2.4.1-misol. Ushbu $e^{k_1x}, e^{k_2x}, \dots, e^{k_nx}$ funksiyalar $k_j \neq k_i, i \neq j$ shart bajarilganda ixtiyoriy $\forall x \in [a, b] \subseteq R$ kesmada chiziqli bog'lanmagan funksiyalar bo'lishini ko'rsating.

Yechish. Faraz qilaylik, bu funksiyalar chiziqli bog'langan bo'lsin, ya'ni

$$a_1e^{k_1x} + a_2e^{k_2x} + \dots + a_ne^{k_nx} = 0$$

tenglik a_j -o'zgarmlarning birortasi, masalan $a_n \neq 0$ bo'lganda o'rinli bo'lsin.

U holda tenglikni e^{k_1x} ga bo'lib

$$a_1 + a_2e^{(k_2-k_1)x} + \dots + a_ne^{(k_n-k_1)x} = 0$$

munosabatni topamiz. Bu tenglikni differensiallab

$$a_2(k_2 - k_1)e^{(k_2-k_1)x} + a_3(k_3 - k_1)e^{(k_3-k_1)x} + \dots + a_n(k_n - k_1)e^{(k_n-k_1)x} = 0$$

munosabatni hosil qilamiz. Bu tenglikni $e^{(k_2-k_1)x}$ ga bo'lib

$$a_2(k_2 - k_1) + a_3(k_3 - k_1)e^{(k_3-k_2)x} + \dots + a_n(k_n - k_1)e^{(k_n-k_2)x} = 0$$

munosabatga ega bo'lamiz. Buni yana differensiallab

$$a_3(k_3 - k_1)(k_3 - k_2)e^{(k_3 - k_2)x} + \dots + a_n(k_n - k_1)(k_n - k_2)e^{(k_n - k_2)x} = 0$$

tenglikka ega bo‘lamiz. Yuqorida bayon qilingan jarayonni davom qildirish natijasida

$$a_n(k_n - k_1)(k_n - k_2) \cdot \dots \cdot (k_n - k_{n-1})e^{(k_n - k_{n-1})x} = 0$$

tenglikka ega bo‘lamiz. Bundan o‘z navbatida

$$e^{(k_n - k_{n-1})x} = 0$$

ekanligi kelib chiqadi. Buning bo‘lishi mumkin emas. Shunday qilib $\{e^{k_1x}, e^{k_2x}, \dots, e^{k_nx}\}$ -funksiyalar sistemasi $k_j \neq k_i, i \neq j$ bo‘lganda chiziqli bog‘lanmagan funksiyalar sistemasini tashkil qilargan ekan.

2.4.2-ta’rif. Ushbu $\varphi_j(x) \in C^{(n-1)}[a, b], j = \overline{1, n}$ funksiyalardan tuzilgan

$$W(x) = W\{\varphi_1, \varphi_2, \dots, \varphi_n\} \equiv \begin{vmatrix} \varphi_1 & \varphi_2 & \dots & \varphi_n \\ \varphi_1' & \varphi_2' & \dots & \varphi_n' \\ \dots & \dots & \dots & \dots \\ \varphi_1^{(n-1)} & \varphi_2^{(n-1)} & \dots & \varphi_n^{(n-1)} \end{vmatrix}$$

determinantga Vronskiy determinanti yoki vronskiyani deyiladi.

2.4.2-teorema. Ushbu

$$L[y] = y^{(n)} + p_1(x)y^{(n-1)} + \dots + p_n(x)y = 0 \quad (2.4.2)$$

bir jinsli differensial tenglamaning $y_1(x), y_2(x), y_3(x), \dots, y_n(x)$ yechimlari chiziqli bog‘liq bo‘lishi uchun, ulardan tuzilgan Vronskiy determinanti nolga teng, ya’ni

$$W\{y_1(x), y_2(x), \dots, y_n(x)\} = 0, \forall x \in [a, b] \quad (2.4.3)$$

bo‘lishi zarur va yetarli.

Isbot. Yetarliligi. Faraz qilaylik (2.4.3) munosabat bajarilsin. U holda $W(x_0) = 0$ bo‘ladigan biror $x_0 \in [a, b]$ nuqtani olib c_1, c_2, \dots, c_n o‘zgarmaslarga nisbatan ushbu

$$\begin{cases} c_1y_1(x_0) + c_2y_2(x_0) + \dots + c_ny_n(x_0) = 0, \\ c_1y_1'(x_0) + c_2y_2'(x_0) + \dots + c_ny_n'(x_0) = 0, \\ \dots \\ c_1y_1^{(n-1)}(x_0) + c_2y_2^{(n-1)}(x_0) + \dots + c_ny_n^{(n-1)}(x_0) = 0 \end{cases} \quad (2.4.4)$$

tenglamalar sistemasini tuzib olamiz. Bu sistemaning asosiy determinanti $W(x_0) = 0$ bo‘lgani uchun bir jinsli (2.4.4) sistema $c_j, j = \overline{1, n}$ -larga nisbatan nolmas yechimga ega. Shuning uchun

$$y(x) = \sum_{j=1}^n c_j y_j(x)$$

funksiya (2.4.2) bir jinsli differensial tenglamaning yechimidan iborat bo‘lib, $x = x_0$ nuqtada

$$y(x_0) = 0, y'(x_0) = 0, \dots, y^{(n-1)}(x_0) = 0 \quad (2.4.5)$$

boshlang'ich shartlarni qanoatlantiradi. Chunki

$$y(x_0) = c_1 y_1(x_0) + c_2 y_2(x_0) + \dots + c_n y_n(x_0) = 0,$$

$$y'(x_0) = c_1 y_1'(x_0) + c_2 y_2'(x_0) + \dots + c_n y_n'(x_0) = 0,$$

.....

$$y^{(n-1)}(x_0) = c_1 y_1^{(n-1)}(x_0) + c_2 y_2^{(n-1)}(x_0) + \dots + c_n y_n^{(n-1)}(x_0) = 0$$

Ikkinchi tomondan $\tilde{y}(x) = 0$ funksiya ham (2.4.2) tenglamani va (2.4.5) boshlang'ich shartni qanoatlantiradi. Yagonalik teoremasiga ko'ra

$$y(x) = \tilde{y}(x) = 0$$

bo'ladi. Bundan

$$\sum_{j=1}^n c_j y_j(x) = 0$$

bo'lishi kelib chiqadi. Bu yerdagi $c_j, j=1, n$ sonlarning kamida bittasi noldan farqli bo'lgani uchun $y_1(x), y_2(x), \dots, y_n(x)$ funksiyalar chiziqli bog'liq bo'ladi.

Zarurligi. Aytaylik, $y_1(x), y_2(x), \dots, y_n(x)$ funksiyalar (2.4.2) bir jinsli differensial tenglamaning yechimlari bo'lsin. U holda ularning chiziqli kombinatsiyasidan tuzilgan

$$y(x) = \sum_{j=1}^n c_j y_j(x), \quad c_j = \text{const}$$

funksiya ham uning yechimi bo'ladi. Teorema shartiga ko'ra $y_1(x), y_2(x), \dots, y_n(x)$ funksiyalar chiziqli bog'liq bo'lgani uchun ushbu

$$c_1 y_1(x) + c_2 y_2(x) + \dots + c_n y_n(x) = 0 \quad (2.4.6)$$

tenglik $c_j, j=1, n$ sonlarning kamida bittasi noldan farqli bo'lganda bajariladi, ya'ni shunday $\exists j_0 \in N$ nomer mavjud bo'lib, $c_{j_0} \neq 0$ bo'ladi. Bu (2.4.6)

tenglikni $(n-1)$ marta differensiallab $c_j, j=1, n$ o'zgarmaslarga nisbatan quyidagi bir jinsli tenglamalar sistemasiga ega bo'lamiz:

$$\begin{cases} c_1 y_1(x) + c_2 y_2(x) + \dots + c_n y_n(x) = 0, \\ c_1 y_1'(x) + c_2 y_2'(x) + \dots + c_n y_n'(x) = 0, \\ \dots \\ c_1 y_1^{(n-1)}(x) + c_2 y_2^{(n-1)}(x) + \dots + c_n y_n^{(n-1)}(x) = 0. \end{cases} \quad (2.4.7)$$

Oxirgi (2.4.7) bir jinsli sistema nolmas c_1, c_2, \dots, c_n yechimga ega. Chunki $c_{j_0} \neq 0$. Shuning uchun (2.4.7) sistemaning asosiy determinanti nolga teng bo'ladi, ya'ni

$$\begin{vmatrix} y_1(x) & y_2(x) & \dots & y_n(x) \\ y_1'(x) & y_2'(x) & \dots & y_n'(x) \\ \dots & \dots & \dots & \dots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \dots & y_n^{(n-1)}(x) \end{vmatrix} = W\{y_1(x), y_2(x), \dots, y_n(x)\} = 0$$

Teorema isbot bo'ldi. ■

2.4.3-teorema. (2.4.2) bir jinsli differensial tenglamaning $y_1(x), y_2(x), \dots, y_n(x)$ yechimlari chiziqli bog'lanmagan bo'lishi uchun ulardan tuzilgan Vronstkiy determinanti

$$W(x) = W\{y_1(x), y_2(x), \dots, y_n(x)\} \neq 0, \quad \forall x \in [a, b]$$

nolmas bo'lishi zarur va yetarli.

Isbot. Zarurligi. Faraz qilaylik (2.4.2) bir jinsli differensial tenglamaning $y_1(x), y_2(x), \dots, y_n(x)$ yechimlari $[a, b]$ oraliqda chiziqli bog'lanmagan bo'lib, biror $x_0 \in (a, b)$ nuqtada

$$W(x_0) = W\{y_1(x), y_2(x), \dots, y_n(x)\}|_{x=x_0} = 0,$$

bo'lsin. Bundan foydalanib c_1, c_2, \dots, c_n o'zgarmlarga nisbatan quyidagi

$$\begin{cases} c_1 y_1(x_0) + c_2 y_2(x_0) + \dots + c_n y_n(x_0) = 0, \\ c_1 y_1'(x_0) + c_2 y_2'(x_0) + \dots + c_n y_n'(x_0) = 0, \\ \dots \\ c_1 y_1^{(n-1)}(x_0) + c_2 y_2^{(n-1)}(x_0) + \dots + c_n y_n^{(n-1)}(x_0) = 0 \end{cases} \quad (2.4.8)$$

tenglamalar sistemasini tuzib olamiz. Bu sistemaning asosiy determinanti $\Delta = W(x_0) = 0$ bo'lgani uchun u nolmas $c_1 = c_1^{(0)}, c_2 = c_2^{(0)}, \dots, c_n = c_n^{(0)}$ yechimga ega. Ushbu $c_1^{(0)}, c_2^{(0)}, \dots, c_n^{(0)}$ sonlarning kamida bittasi noldan farqli. Natijada ushbu

$$y(x) = c_1^{(0)} y_1(x) + c_2^{(0)} y_2(x) + \dots + c_n^{(0)} y_n(x) \quad (2.4.9)$$

funksiya (2.4.2) bir jinsli differensial tenglamani va

$$y(x_0) = 0, y'(x_0) = 0, \dots, y^{(n-1)}(x_0) = 0 \quad (2.4.10)$$

boshlang'ich shartlarni qanoatlantiradi. Oldingi paragrafdagi 2.3.1-natijaga ko'ra, (2.4.2) bir jinsli differensial tenglamaga qo'yilgan (2.4.10) boshlang'ich masala faqat $y(x) \equiv 0$ nol yechimga ega ekanligidan (2.4.9) tenglik ushbu

$$c_1^{(0)} y_1(x) + c_2^{(0)} y_2(x) + \dots + c_n^{(0)} y_n(x) = 0, \quad x \in (a, b)$$

ko'rinishni oladi. Bunda $c_j^{(0)}, j = \overline{1, n}$ sonlar orasida kamida bittasi noldan farqli. Shuning uchun (2.4.2) differensial tenglamaning $y_1(x), y_2(x), \dots, y_n(x)$ yechimlari chiziqli bog'langan. Bu esa $y_1(x), y_2(x), \dots, y_n(x)$ yechimlarning chiziqli bog'lanmaganligiga zid.

Yetarliligi. Aytaylik, $W(x) \neq 0, x \in (a, b)$ bo'lib, $y_1(x), y_2(x), \dots, y_n(x)$ yechimlar chiziqli bog'liq bo'lsin. U holda

$$\alpha_1 y_1(x) + \alpha_2 y_2(x) + \dots + \alpha_n y_n(x) = 0, \quad x \in (a, b)$$

ayniyat o'rinli. Bu yerda, masalan $\alpha_n \neq 0$. Bundan

$$y_n(x) = -\frac{\alpha_1}{\alpha_n} y_1(x) - \frac{\alpha_2}{\alpha_n} y_2(x) - \dots - \frac{\alpha_{n-1}}{\alpha_n} y_{n-1}(x)$$

kelib chiqadi. Bu tenglikni differensiallab

$$y_n^{(j)}(x) = -\frac{\alpha_1}{\alpha_n} y_1^{(j)}(x) - \frac{\alpha_2}{\alpha_n} y_2^{(j)}(x) - \dots - \frac{\alpha_{n-1}}{\alpha_n} y_{n-1}^{(j)}(x), \quad j = \overline{0, n-1}$$

topamiz. O'z navbatida bu tengliklardan $W(x) = 0$ kelib chiqadi. Bu esa farazimizga zid. Teorema isbot bo'ldi. ■

Mustaqil yechish uchun mashqlar [8], §15, №371-386, №389-400; [21], §12, №601-622.

5-§. n-tartibli chiziqli bir jinsli differensial tenglamaning fundamental yechimlar sistemasi

2.5.1-ta'rif. Ushbu

$$L[y] = y^{(n)} + p_1(x)y^{(n-1)} + \dots + p_n(x)y = 0 \quad (2.5.1)$$

differensial tenglamaning ixtiyoriy n ta $y_1(x), y_2(x), \dots, y_n(x)$ chiziqli bo'g'lanmagan yechimlariga, uning fundamental yechimlari sistemasi (F.Y.S) deyiladi.

2.5.1-teorema. Koeffitsiyentlari uzluksiz $p_j(x) \in C[a, b], j = \overline{1, n}$ funksiyalardan iborat bo'lgan (2.5.1) ko'rinishdagi bir jinsli differensial tenglamaning fundamental yechimlari sistemasi (F.S.Y) mavjud.

Isbot. Aytaylik, $a_{ij} \in R, j, i = \overline{1, n}$ sonlardan tuzilgan

$$\Delta^0 = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} \neq 0$$

determinant nolga teng bo'lmasin. U holda (2.5.1) differensial tenglamaning ushbu

$$y_j(x_0) = a_{1j}, y_j'(x_0) = a_{2j}, \dots, y_j^{(n-1)}(x_0) = a_{nj}, \quad j = \overline{1, n} \quad (2.5.2)$$

boshlang'ich shartlarni qanoatlantiruvchi $y_1(x), y_2(x), \dots, y_n(x)$ yechimlari mavjud. Bu yechimlardan tuzilgan Vronskiy determinantini qaraylik:

$$W(x) = W\{y_1, y_2, \dots, y_n\}.$$

Endi $W(x_0)$ ni hisoblaymiz:

$$W(x_0) = \begin{vmatrix} y_1(x_0) & y_2(x_0) & \dots & y_n(x_0) \\ y_1'(x_0) & y_2'(x_0) & \dots & y_n'(x_0) \\ \dots & \dots & \dots & \dots \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \dots & y_n^{(n-1)}(x_0) \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} \neq 0.$$

Shuning uchun $y_1(x), y_2(x), \dots, y_n(x)$ funksiyalar chiziqli bog'lanmagan bo'ladi. Demak (2.5.1) bir jinsli differensial tenglamaning $F.Y.S = \{y_1(x), y_2(x), \dots, y_n(x)\}$ mavjud ekan. ■

Izoh. Noldan farqli $\Delta^0 \neq 0$ determinantlar cheksiz ko'p bo'lgani uchun (2.5.1) ko'rinishdagi bir jinsli differensial tenglamaning F.Y.S ham cheksiz ko'p bo'ladi.

2.5.2-teorema. Agar $y_1(x), y_2(x), \dots, y_n(x)$ funksiyalar (2.5.1) bir jinsli differensial tenglamaning F.Y.S ni tashkil qilsa, u holda uning umumiy yechimi ushbu

$$y(x) = \sum_{j=1}^n c_j y_j(x), \quad \forall c_j = const \quad (2.5.3)$$

ko'rinishda yoziladi.

Isbot. (2.5.3) ko'rinishdagi $y(x)$ funksiya ushbu

$$G = \{a < x < b, |y| < \infty, \dots, |y^{(n-1)}| < \infty\}$$

sohada (2.5.1) tenglamaning umumiy yechimi ekanligini ko'rsatamiz. Chunki G sohaning har bir nuqtasida Koshi teoremasining shartlari bajariladi.

1. Quyidagi

$$\begin{aligned} y(x) &= \sum_{j=1}^n c_j y_j(x), \\ y'(x) &= \sum_{j=1}^n c_j y_j'(x), \\ &\dots \\ y^{(n-1)}(x) &= \sum_{j=1}^n c_j y_j^{(n-1)}(x), \end{aligned} \quad (2.5.4)$$

algebraik tenglamalar sistemasi c_1, c_2, \dots, c_n -ixtiyoriy o'zgaraslarga nisbatan yechimga ega. Chunki, bu sistemaning asosiy determinant noldan farqli, ya'ni

$$\Delta = W(x) = W\{y_1(x), y_2(x), \dots, y_n(x)\} \neq 0, \quad x \in (a, b).$$

2. c_1, c_2, \dots, c_n -o'zgaraslarning ixtiyoriy qiymatlarida (2.5.3) tenglik orqali aniqlangan $y(x)$ funksiya (2.5.1) bir jinsli differensial tenglamaning yechimidan

iborat. Shuning uchun (2.5.3) tenglik orqali aniqlangan $y(x)$ funksiya G sohada (2.5.1) bir jinsli differensial tenglamaning umumiy yechimi bo‘ladi.

Yuqoridagi (2.5.3) formula (2.5.1) differensial tenglamaning barcha yechimlarini o‘z ichiga oladi. Jumladan ushbu

$$L[y]=0, \quad y(x_0) = y_0, \quad y'(x_0) = y'_0, \dots, y^{(n-1)}(x_0) = y_0^{(n-1)} \quad (2.5.5)$$

Koshi masalasining yechimi ham (2.5.3) formula tarkibiga kiradi. Bunda $y_0, y'_0, \dots, y_0^{(n-1)}$ va $x = x_0$ ixtiyoriy berilgan sonlar. Yuqoridagi (2.5.4) sistemani (2.5.5) dan foydalanib quyidagicha yozish mumkin:

$$\begin{cases} y_0 = \sum_{j=1}^n c_j y_j(x_0), \\ y'_0 = \sum_{j=1}^n c_j y'_j(x_0), \\ \dots \dots \dots \\ y_0^{(n-1)} = \sum_{j=1}^n c_j y_j^{(n-1)}(x_0). \end{cases} \quad (2.5.6)$$

Bu sistemaning asosiy determinanti $\Delta = W(x_0) \neq 0$ noldan farqli bo‘lgani uchun, u yagona $c_1 = c_1^{(0)}, c_2 = c_2^{(0)}, \dots, c_n = c_n^{(0)}$ yechimga ega. Bu topilgan $C_j = C_j^{(0)}, j = \overline{1, n}$ larni (2.5.3) formulaga qo‘yib

$$y(x) = \sum_{j=1}^n c_j^{(0)} y_j(x),$$

izlanayotgan (2.5.5) Koshi masalasining yechimini topamiz. Shuning uchun (2.5.1) tenglamaning $F.Y.S = \{y_1(x), y_2(x), \dots, y_n(x)\}$ yechimlar fazosining bazasini tashkil qiladi. $L[y]=0$ tenglama yechimlari fazosi n -o‘lchamli chiziqli fazo bo‘ladi.

6-§. n -tartibli bir jinsli differensial tenglamani fundamental yechimlar sistemasi yordamida aniqlash

2.6.1-teorema. Agar ikkita

$$y^{(n)} + p_1(x)y^{(n-1)} + \dots + p_{n-1}(x)y' + p_n(x)y = 0 \quad (2.6.1)$$

$$y^{(n)} + \tilde{p}_1(x)y^{(n-1)} + \dots + \tilde{p}_{n-1}(x)y' + \tilde{p}_n(x)y = 0 \quad (2.6.2)$$

bir jinsli differensial tenglamalar umumiy F.Y.S ga ega bo‘lsa, u holda

$$p_1(x) = \tilde{p}_1(x), p_2(x) = \tilde{p}_2(x), \dots, p_n(x) = \tilde{p}_n(x), \quad \forall x \in (a, b) \quad (2.6.3)$$

munosabatlar o‘rinli bo‘ladi. Bu yerda $p_j(x), \tilde{p}_j(x) \in C[a, b]$ -uzluksiz funksiyalar.

Isbot. Faraz qilaylik, (2.6.1) va (2.6.2) differensial tenglamalarning F.Y.S $\{y_1(x), y_2(x), \dots, y_n(x)\}$ -funksiyalardan iborat bo'lsin. U holda (2.6.1) tenglikdan (2.6.2) ni ayirib quyidagi

$$[p_1(x) - \tilde{p}_1(x)]y^{(n-1)} + \dots + [p_{n-1}(x) - \tilde{p}_{n-1}(x)]y' + [p_n(x) - \tilde{p}_n(x)]y = 0 \quad (2.6.4)$$

differensial tenglamani hosil qilamiz. Avvalo, aniqlik uchun $p_1(x) \neq \tilde{p}_1(x)$ deylik. U holda shunday $\exists (\alpha, \beta) \subset (a, b)$ interval topilib, $p_1(x) \neq \tilde{p}_1(x)$, $\forall x \in (\alpha, \beta)$ o'rinli bo'ladi. So'ngra (2.6.4) tenglikning ikki tomoni $p_1(x) - \tilde{p}_1(x) \neq 0$, $\forall x \in (\alpha, \beta)$ ga bo'lib

$$y^{(n-1)} + d_2(x)y^{(n-2)} + \dots + d_n(x)y = 0 \quad (2.6.5)$$

differensial tenglamaga ega bo'lamiz. Bu differensial tenglama uchun $\{y_1(x), y_2(x), \dots, y_n(x)\}$ -funksiyalar yechim bo'ladi. Lekin (2.6.5) differensial tenglama $n-1$ -tartibli bo'lgani uchun $y_1(x), y_2(x), \dots, y_n(x)$ funksiyalar (α, β) intervalda chiziqli bog'langan bo'ladi. Bu qarama-qarshilik, $p_1(x) = \tilde{p}_1(x)$, $\forall x \in (\alpha, \beta)$ ekanligini bildiradi. Xuddi shunday mulohaza yurgizish orqali

$$p_j(x) = \tilde{p}_j(x), \quad j \geq 2$$

tenglik ham isbotlanadi. Shunday qilib fundamental yechimlar sistemasi (F.Y.S.) bosh koeffitsiyenti 1 ga teng bo'lgan bir jinsli chiziqli differensial tenglamani yagona aniqlaydi. ■

Endi (2.6.1) differensial tenglamani, uning fundamental yechimlari orqali qurish mumkinligini bayon qilamiz. Aytaylik, ushbu $y_1(x), y_2(x), \dots, y_n(x)$ funksiyalar (2.6.1) differensial tenglamaning fundamental yechimlaridan iborat bo'lsin, ya'ni

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) & \dots & y_n(x) \\ y_1'(x) & y_2'(x) & \dots & y_n'(x) \\ \dots & \dots & \dots & \dots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \dots & y_n^{(n-1)}(x) \end{vmatrix} \neq 0.$$

Aytaylik, $y=y(x)$ funksiya biz izlayotgan bir jinsli n -tartibli chiziqli differensial tenglamaning ixtiyoriy yechimi bo'lsin. U holda ushbu

$$W\{y(x), y_1(x), \dots, y_n(x)\} = 0$$

tenglik o'rinli bo'ladi, ya'ni

$$\begin{vmatrix} y_1(x) & y_2(x) & \dots & y_n(x) & y(x) \\ y_1'(x) & y_2'(x) & \dots & y_n'(x) & y'(x) \\ \dots & \dots & \dots & \dots & \dots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \dots & y_n^{(n-1)}(x) & y^{(n-1)}(x) \\ y_1^{(n)}(x) & y_2^{(n)}(x) & \dots & y_n^{(n)}(x) & y^{(n)}(x) \end{vmatrix} = 0. \quad (2.6.6)$$

Berilgan $y_1(x), y_2(x), \dots, y_n(x)$ funksiyalarni birin ketin (2.6.6) tenglikdagi $y(x)$ o'rniga qo'yilsa, bu tenglama ayniyatga aylanadi. Bundan ko'rinadiki, $y_j(x), j = \overline{1, n}$ funksiyalar (2.6.6) tenglamaning yechimlaridan iborat bo'ladi.

Endi (2.6.6) determinantni $n + 1$ -ustun bo'yicha yoyamiz:

$$\begin{aligned}
 & \begin{vmatrix} y_1(x) & y_2(x) & \dots & y_n(x) \\ y_1'(x) & y_2'(x) & \dots & y_n'(x) \\ \dots & \dots & \dots & \dots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \dots & y_n^{(n-1)}(x) \end{vmatrix} y^{(n)} - \\
 & - \begin{vmatrix} y_1(x) & y_2(x) & \dots & y_n(x) \\ y_1'(x) & y_2'(x) & \dots & y_n'(x) \\ \dots & \dots & \dots & \dots \\ y_1^{(n-2)}(x) & y_2^{(n-2)}(x) & \dots & y_n^{(n-2)}(x) \\ y_1^{(n)}(x) & y_2^{(n)}(x) & \dots & y_n^{(n)}(x) \end{vmatrix} y^{(n-1)} + \\
 & + \dots + (-1)^n \begin{vmatrix} y_1'(x) & y_2'(x) & \dots & y_n'(x) \\ y_1''(x) & y_2''(x) & \dots & y_n''(x) \\ \dots & \dots & \dots & \dots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \dots & y_n^{(n-1)}(x) \\ y_1^{(n)}(x) & y_2^{(n)}(x) & \dots & y_n^{(n)}(x) \end{vmatrix} y = 0
 \end{aligned}$$

Bu tenglikning ikki tomonini quyidagi

$$\begin{vmatrix} y_1(x) & y_2(x) & \dots & y_n(x) \\ y_1'(x) & y_2'(x) & \dots & y_n'(x) \\ \dots & \dots & \dots & \dots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \dots & y_n^{(n-1)}(x) \end{vmatrix} = W(x) \neq 0$$

determinantga bo'lib, ushbu

$$y^{(n)} + p_1(x)y^{(n-1)} + \dots + p_n(x)y = 0$$

differensial tenglamaga ega bo'lamiz.

Bu yerda

$$p_1(x) = \frac{-1}{W(x)} \begin{vmatrix} y_1(x) & y_2(x) & \dots & y_n(x) \\ y_1'(x) & y_2'(x) & \dots & y_n'(x) \\ \dots & \dots & \dots & \dots \\ y_1^{(n-2)}(x) & y_2^{(n-2)}(x) & \dots & y_n^{(n-2)}(x) \\ y_1^{(n)}(x) & y_2^{(n)}(x) & \dots & y_n^{(n)}(x) \end{vmatrix},$$

$$p_n(x) = \frac{(-1)^n}{W(x)} \begin{vmatrix} y_1'(x) & y_2'(x) & \dots & y_n'(x) \\ y_1''(x) & y_2''(x) & \dots & y_n''(x) \\ \dots & \dots & \dots & \dots \\ y_1^{(n-1)}(x) & y_2^{(n-2)}(x) & \dots & y_n^{(n-2)}(x) \\ y_1^{(n)}(x) & y_2^{(n)}(x) & \dots & y_n^{(n)}(x) \end{vmatrix}. \quad (2.6.7)$$

Mustaqil yechish uchun mashqlar [8], §15, №420-431, №389-400; [21], §12, №624-630.

7-§. Ostrogradskiy-Liuvill formulasi

Ushbu $y_1(x), y_2(x), \dots, y_n(x)$ funksiyalar quyidagi

$$y^{(n)} + p_1(x)y^{(n-1)} + \dots + p_{n-1}(x)y' + p_n(x)y = 0 \quad (2.7.1)$$

bir jinsli chiziqli differensial tenglamaning fundamental yechimlari sistemasini (F.Y.S) tashkil qilsin.

Bunda quyidagi

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) & \dots & y_n(x) \\ y_1'(x) & y_2'(x) & \dots & y_n'(x) \\ \dots & \dots & \dots & \dots \\ y_1^{(n-2)}(x) & y_2^{(n-2)}(x) & \dots & y_n^{(n-2)}(x) \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \dots & y_n^{(n-1)}(x) \end{vmatrix} \neq 0 \quad (2.7.2)$$

munosabat o‘rinli. Endi Vronskiy determinantining hosilasini hisoblaymiz:

$$W'(x) = \frac{d}{dx}W(x) = \begin{vmatrix} y_1'(x) & y_2'(x) & \dots & y_n'(x) \\ y_1'(x) & y_2'(x) & \dots & y_n'(x) \\ \dots & \dots & \dots & \dots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \dots & y_n^{(n-1)}(x) \end{vmatrix} +$$

$$+ \begin{vmatrix} y_1(x) & y_2(x) & \dots & y_n(x) \\ y_1''(x) & y_2''(x) & \dots & y_n''(x) \\ y_1''(x) & y_2''(x) & \dots & y_n''(x) \\ \dots & \dots & \dots & \dots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \dots & y_n^{(n-1)}(x) \end{vmatrix} + \dots +$$

$$\begin{vmatrix} y_1(x) & y_2(x) & \dots & y_n(x) \\ y_1'(x) & y_2'(x) & \dots & y_n'(x) \\ \dots & \dots & \dots & \dots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \dots & y_n^{(n-1)}(x) \end{vmatrix} +$$

$$\begin{vmatrix} y_1(x) & y_2(x) & \dots & y_n(x) \\ y_1'(x) & y_2'(x) & \dots & y_n'(x) \\ \dots & \dots & \dots & \dots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \dots & y_n^{(n-1)}(x) \end{vmatrix}$$

$$+ \begin{vmatrix} y_1(x) & y_2(x) & \dots & y_n(x) \\ y_1'(x) & y_2'(x) & \dots & y_n'(x) \\ \dots & \dots & \dots & \dots \\ y_1^{(n-2)}(x) & y_2^{(n-2)}(x) & \dots & y_n^{(n-2)}(x) \\ y_1^{(n)}(x) & y_2^{(n)}(x) & \dots & y_n^{(n)}(x) \end{vmatrix}.$$

Bu tenglikda oxirgi determinantdan tashqari barcha determinantlarning qiymati nolga teng. Chunki ularning har birida ikkita satr elementlari bir xil. Shuning uchun oxirgi tenglik quyidagi ko‘rinishni oladi:

$$W'(x) = \begin{vmatrix} y_1(x) & y_2(x) & \dots & y_n(x) \\ y_1'(x) & y_2'(x) & \dots & y_n'(x) \\ \dots & \dots & \dots & \dots \\ y_1^{(n-2)}(x) & y_2^{(n-2)}(x) & \dots & y_n^{(n-2)}(x) \\ y_1^{(n)}(x) & y_2^{(n)}(x) & \dots & y_n^{(n)}(x) \end{vmatrix}. \quad (2.7.3)$$

Bu yerda oldingi paragrafdagi ushbu

$$p_1(x) = \frac{-1}{w(x)} \begin{vmatrix} y_1(x) & y_2(x) & \dots & y_n(x) \\ y_1'(x) & y_2'(x) & \dots & y_n'(x) \\ \dots & \dots & \dots & \dots \\ y_1^{(n-2)}(x) & y_2^{(n-2)}(x) & \dots & y_n^{(n-2)}(x) \\ y_1^{(n)}(x) & y_2^{(n)}(x) & \dots & y_n^{(n)}(x) \end{vmatrix}$$

formulani inobatga olsak, (2.7.3) tenglik quyidagi

$$p_1(x) = -\frac{W'(x)}{W(x)} \quad (2.7.4)$$

ko‘rinishni oladi. Bu esa o‘zgaruvchilari ajraladigan differensial tenglamadir. Oxirgi (2.7.4) tenglikni integrallab

$$W(x) = W(x_0) \exp \left\{ -\int_{x_0}^x p_1(t) dt \right\} \quad (2.7.5)$$

Ostrogradskiy-Liuvill formulasini hosil qilamiz.

Agar biror $x_0 \in (a, b)$ nuqtada $W(x_0) = 0$ bo‘lsa, u holda (2.7.5) formuladan

$$W(x) \equiv 0, \quad \forall x \in (a, b)$$

bo‘lishi kelib chiqadi. Agar biror $x_0 \in (a, b)$ nuqtada $W(x_0) \neq 0$ bo‘lsa, u holda (2.7.5) formuladan

$$W(x) \neq 0, \quad \forall x \in (a, b)$$

ekanligi kelib chiqadi.

8-§. Ostragradskiy- Liuvill formulasi (n=2 bo'lgan hol)

Aytaylik, ushbu

$$y'' + p_1(x)y' + p_2(x)y = 0 \quad (2.8.1)$$

ikkinchi tartibli bir jinsli chiziqli differensial tenglamaning bitta nolmas $0 \neq y_1(x)$ -xususiy yechimi berilgan bo'lsin. U holda (2.8.1) tenglamaning $y_1(x)$ dan farqli, lekin unga chiziqli bog'lanmagan ikkinchi $y_2(x)$ yechimini topish mumkinligini bayon qilamiz. Faraz qilaylik (2.8.1) differensial tenglamaning $y_1(x)$ dan farqli ixtiyoriy yechimi $y(x)$ bo'lsin. U holda $y(x)$ va $y_1(x)$ yechimlardan tuzilgan Vronskiy determinanti

$$W(x) = \begin{vmatrix} y_1(x) & y(x) \\ y_1'(x) & y'(x) \end{vmatrix}$$

uchun quyidagi

$$W(x) = W(x_0) \exp\left\{-\int p_1(x)dx\right\}, \quad c = W(x_0) \quad (2.8.2)$$

Ostragradskiy-Liuvill formulasi o'rinli. Endi (2.8.2) tenglikni ushbu

$$y_1(x)y'(x) - y_1'(x)y(x) = c \exp\left\{-\int p_1(x)dx\right\} \quad (2.8.3)$$

ko'rinishda yozib uni ikki tamonini $\frac{1}{y_1^2}$ ga ko'paytirib

$$\frac{d}{dx}\left(\frac{y(x)}{y_1(x)}\right) = c \frac{1}{y_1^2(x)} e^{-\int p_1(x)dx} \quad (2.8.4)$$

differensial tenglamani hosil qilamiz. Oxirgi (2.8.4) differensial tenglamani integrallab

$$y(x) = c_1 y_1(x) + c y_1(x) \int \frac{1}{y_1^2(x)} e^{-\int p_1(x)dx} dx \quad (2.8.5)$$

formulani topamiz. Bu tenglik orqali aniqlangan $y(x)$ funksiya (2.8.1) differensial tenglamaning umumiy yechimini ifodalaydi.

Ushbu

$$y_2(x) = y_1(x) \int \frac{1}{y_1^2(x)} e^{-\int p_1(x)dx} dx \quad (2.8.6)$$

funksiya (2.8.1) differensial tenglamaning $y_1(x)$ dan farqli, unga chiziqli bog'lanmagan yechimini ifodalaydi.

Agar (2.8.1) differensial tenglamada $p_1(x) = 0$, ya'ni

$$y'' + p_2(x)y = 0 \quad (2.8.7)$$

bo'lsa, u holda uning umumiy yechimi

$$y(x) = c_1 y_1(x) + c y_1(x) \int \frac{1}{y_1^2(x)} dx \quad (2.8.8)$$

ko‘rinishda bo‘ladi. Quyidagi

$$y_2(x) = y_1(x) \int \frac{1}{y_1^2(x)} dx$$

funksiya (2.8.7) tenglamaning $y_1(x)$ dan farqli, lekin unga chiziqli bog‘lanmagan yechimini ifodalaydi. Haqiqatan ham:

$$W\{y_1(x), y_2(x)\} = \begin{vmatrix} y_1 & y_1 \int \frac{1}{y_1^2(x)} dx \\ y_1' & y_1' \int \frac{dx}{y_1^2(x)} + \frac{1}{y_1(x)} \end{vmatrix} = 1.$$

Ayrim adabiyotlarda (2.8.8) formulaga Abel formulasi deb ham yuritiladi.

Mustaqil yechish uchun mashqlar [21], §12, №631-640; [8], §15, №636-648.

9-§. Abel formulasi umumlashmasi

2.9.1-teorema. Aytaylik $y_1(x), y_2(x), \dots, y_{n-1}(x)$ funksiyalar $n \geq 2$ - tartibli bir jinsli

$$L[y] \equiv y^{(n)} + p_1(x)y^{(n-1)} + \dots + p_{n-1}(x)y' + p_n(x)y = 0 \quad (2.9.1)$$

chiziqli differensial tenglamaning chiziqli bog‘lanmagan yechimlari bo‘lib,

$$W(x) = W\{y_1(x), y_2(x), \dots, y_{n-1}(x)\} \neq 0, \quad x \in I = [a, b]$$

bo‘lsin. U holda (2.9.1) differensial tenglamaning umumiy yechimi

$$y(x) = \sum_{j=1}^{n-1} c_j y_j(x) + c_n \int_{x_0}^x \frac{K_{n-1}(x, \tau)}{W^2(\tau)} e^{-\int_{x_0}^{\tau} p_1(t) dt} d\tau, \quad c_j = \text{const}, \quad x_0, x \in I \quad (2.9.2)$$

ko‘rinishda bo‘ladi. Bunda $p_j(x) \in C[a, b]$, $K_1(x, \tau) = y_1(x)$.

$$K_{n-1}(x, \tau) = \begin{vmatrix} y_1(\tau) & y_2(\tau) & \dots & y_{n-1}(\tau) \\ y_1'(\tau) & y_2'(\tau) & \dots & y_{n-1}'(\tau) \\ \dots & \dots & \dots & \dots \\ y_1^{(n-3)}(\tau) & y_2^{(n-3)}(\tau) & \dots & y_{n-1}^{(n-3)}(\tau) \\ y_1(x) & y_2(x) & \dots & y_{n-1}(x) \end{vmatrix}, \quad n \geq 3.$$

Isbot. Yuqoridagi ikkinchi formulaning o‘rinli ekanligini ko‘rsatish uchun

$$y_n(x) = \int_{x_0}^x \frac{K_{n-1}(x, \tau)}{W^2(\tau)} e^{-\int_{x_0}^{\tau} p_1(t) dt} d\tau, \quad (x, x_0 \in I = (a, b)) \quad (2.9.3)$$

funksiyani (2.9.1) differensial tenglamaning yechimi ekanligini ko'rsatish yetarli. $K_{n-1}(x, \tau)$ -determinantni oxirgi satr elementlari bo'yicha yoyish natijasida

$$L[K_{n-1}(x, \tau)] = 0, \forall \tau \in I \quad (2.9.4)$$

tenglamaga ega bo'lamiz. Haqiqatan ham, ushbu

$$K_{n-1}(x, \tau) = (-1)^n \begin{vmatrix} y_2(\tau) & y_3(\tau) & \dots & y_{n-1}(\tau) \\ y_2'(\tau) & y_3'(\tau) & \dots & y_{n-1}'(\tau) \\ \dots & \dots & \dots & \dots \\ y_2^{(n-3)}(\tau) & y_3^{(n-3)}(\tau) & \dots & y_{n-1}^{(n-3)}(\tau) \end{vmatrix} y_1(x) +$$

$$+ (-1)^{n+1} \begin{vmatrix} y_1(\tau) & y_3(\tau) & \dots & y_{n-1}(\tau) \\ y_1'(\tau) & y_3'(\tau) & \dots & y_{n-1}'(\tau) \\ \dots & \dots & \dots & \dots \\ y_1^{(n-3)}(\tau) & y_3^{(n-3)}(\tau) & \dots & y_{n-1}^{(n-3)}(\tau) \end{vmatrix} y_2(x) + \dots +$$

$$+ \begin{vmatrix} y_1(\tau) & y_2(\tau) & \dots & y_{n-2}(\tau) \\ y_1'(\tau) & y_2'(\tau) & \dots & y_{n-2}'(\tau) \\ \dots & \dots & \dots & \dots \\ y_1^{(n-3)}(\tau) & y_2^{(n-3)}(\tau) & \dots & y_{n-2}^{(n-3)}(\tau) \end{vmatrix} y_{n-1}(x)$$

yoyilmaning ikki tamoniga L operatorni qo'llasak, (2.9.4) differensial tenglama kelib chiqadi. Bu esa $K_{n-1}(x, \tau)$ funksiya (2.9.1) differensial tenglamaning yechimi ekanligini bildiradi. Yuqoridagi (2.9.3) tenglik bilan aniqlangan $y_n(x)$ funksiyaning hosilasini hisoblash qiyinchilik tug'dirmaydi.

$$y_n'(x) = \int_{x_0}^x \frac{K_{n-1}'(x, \tau)}{W^2(\tau)} e^{-\int_{x_0}^{\tau} p_1(t) dt} d\tau,$$

.....

$$y_n^{(n-2)}(x) = \int_{x_0}^x \frac{K_{n-1}^{(n-2)}(x, \tau)}{W^2(\tau)} e^{-\int_{x_0}^{\tau} p_1(t) dt} d\tau,$$

$$y_n^{(n-1)}(x) = \int_{x_0}^x \frac{K_{n-1}^{(n-1)}(x, \tau)}{W^2(\tau)} e^{-\int_{x_0}^{\tau} p_1(t) dt} d\tau + \frac{1}{W(x)} e^{-\int_{x_0}^x p_1(x) dx},$$

$$y_n^{(n)}(x) = \int_{x_0}^x \frac{K_{n-1}^{(n)}(x, \tau)}{W^2(\tau)} e^{-\int_{x_0}^{\tau} p_1(t) dt} d\tau - \frac{p_1(x)}{W(x)} e^{-\int_{x_0}^x p_1(t) dt}, \quad (2.9.5)$$

(2.9.3) va (2.9.5) munosabatlardan foydalanib $L[y_n(x)]$ ning qiymatini topamiz:

$$L[y_n(x)] \equiv y_n^{(n)}(x) + p_1(x)y_n^{(n-1)}(x) + \dots + p_{n-1}(x)y_n'(x) + p_n(x)y_n(x) =$$

$$= \int_{x_0}^x \frac{L[K_{n-1}(x, \tau)]}{W^2(\tau)} e^{-\int_{x_0}^{\tau} p_1(t)dt} d\tau - \frac{p_1(x)}{W(x)} e^{-\int_{x_0}^x p_1(t)dt} + \frac{p_1(x)}{W(x)} e^{-\int_{x_0}^x p_1(\tau)d\tau} = 0, \quad \forall x \in I = (a, b)$$

(2.9.3) va (2.9.5) tengliklardan ushbu

$$y_n(x_0) = y_n'(x_0) = \dots = y_n^{(n-2)}(x_0) = 0, \quad y_n^{(n-1)}(x_0) = [W(x_0)]^{-1}$$

boshlang'ich shart kelib chiqadi. Bundan foydalanib, $W\{y_1(x), y_2(x), \dots, y_n(x)\}|_{x=x_0}$ ifodaning qiymatini topamiz:

$$W\{y_1(x), y_2(x), \dots, y_n(x)\}|_{x=x_0} =$$

$$= \begin{vmatrix} y_1(x_0) & y_2(x_0) & \dots & y_{n-1}(x_0) & 0 \\ y_1'(x_0) & y_2'(x_0) & \dots & y_{n-1}'(x_0) & 0 \\ \dots & \dots & \dots & \dots & \dots \\ y_1^{(n-2)}(x_0) & y_2^{(n-2)}(x_0) & \dots & y_{n-1}^{(n-2)}(x_0) & 0 \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \dots & y_{n-1}^{(n-1)}(x_0) & [W(x_0)]^{-1} \end{vmatrix} = 1.$$

Bundan ko'rinadiki, $y_1(x), y_2(x), \dots, y_n(x)$ funksiyalar (2.9.1) differensial tenglamaning fundamental yechimlari sistemasini tashkil qiladi. Shuning uchun (2.9.2) formula (2.9.1) differensial tenglamaning umumiy yechimini beradi.

2.9.2-teorema. Faraz qilaylik, $y_1(x), y_2(x), \dots, y_n(x)$ funksiyalar (2.9.1) differensial tenglamaning fundamental yechimlari sistemasidan iborat bo'lsin. U holda quyidagi

$$z_k(x) = \sum_{j=1}^n a_{kj} y_j(x), \quad k = \overline{1, n} \quad a_{kj} = const \quad (2.9.6)$$

funksiyalar (2.9.1) differensial tenglamaning fundamental yechimlari sistemasidan iborat bo'lishi uchun

$$\Delta = \det(a_{kj}) \neq 0, \quad k, j = \overline{1, n} \quad (2.9.7)$$

bo'lishi zarur va yetarli.

Isbot. L operatorning chiziqliligidan va (2.9.6) tenglikdan

$$L[z_k(x)] = L\left[\sum_{j=1}^n a_{kj} y_j(x)\right] = \sum_{j=1}^n a_{kj} L[y_j(x)] = 0$$

kelib chiqadi. Bu esa $z_1(x), z_2(x), \dots, z_n(x)$ funksiyalar (2.9.1) differensial tenglamaning yechimlaridan iborat ekanligini ko'rsatadi. Shuning uchun, bu $z_1(x), z_2(x), \dots, z_n(x)$ funksiyalardan tuzilgan Vronskiy determinanti

$$W\{z_1(x), z_2(x), \dots, z_n(x)\} \neq 0, \forall x \in I$$

noldan farqli bo'lishi uchun (2.9.7) shartning bajarilishi zarur va yetarli ekanligini ko'rsatamiz. Shu maqsadda $z_1(x), z_2(x), \dots, z_n(x)$ funksiyalardan tuzilgan Vronskiy determinantini (2.9.6) tenglikdan foydalanib hisoblaymiz:

$$\begin{aligned} W\{z_1(x), z_2(x), \dots, z_n(x)\} &= \begin{vmatrix} z_1(x) & z_2(x) & \dots & z_n(x) \\ z_1'(x) & z_2'(x) & \dots & z_n'(x) \\ \dots & \dots & \dots & \dots \\ z_1^{(n-1)}(x) & z_2^{(n-1)}(x) & \dots & z_n^{(n-1)}(x) \end{vmatrix} = \\ &= \begin{vmatrix} \sum_{j=1}^n a_{1j} y_j(x) & \sum_{j=1}^n a_{2j} y_j(x) & \dots & \sum_{j=1}^n a_{nj} y_j(x) \\ \sum_{j=1}^n a_{1j} y_j'(x) & \sum_{j=1}^n a_{2j} y_j'(x) & \dots & \sum_{j=1}^n a_{nj} y_j'(x) \\ \dots & \dots & \dots & \dots \\ \sum_{j=1}^n a_{1j} y_j^{(n-1)}(x) & \sum_{j=1}^n a_{2j} y_j^{(n-1)}(x) & \dots & \sum_{j=1}^n a_{nj} y_j^{(n-1)}(x) \end{vmatrix} = \\ &= \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} y_1(x) & y_1'(x) & \dots & y_1^{(n-1)}(x) \\ y_2(x) & y_2'(x) & \dots & y_2^{(n-1)}(x) \\ \dots & \dots & \dots & \dots \\ y_n(x) & y_n'(x) & \dots & y_n^{(n-1)}(x) \end{pmatrix} = \\ &= \det(a_{ij}) W\{y_1(x), y_2(x), \dots, y_n(x)\} = \Delta \cdot W(x) \end{aligned}$$

Bu tenglikda $W(x) \neq 0, x \in I$ ekanligini e'tiborga olsak $W\{z_1(x), z_2(x), \dots, z_n(x)\} \neq 0$ bo'lishi uchun $\Delta = \det(a_{ij}) \neq 0$ bo'lishi zarur va yetarli. ■

10-§. n-tartibli bir jinsli o'zgarmas koeffitsiyentli chiziqli differensial tenglama

Agar bir jinsli differensial tenglamaning fundamental yechimlari sistemasi (F.Y.S.) ma'lum bo'lsa, u holda uning ixtiyoriy yechimini topish mumkin.

Beshinchi paragrafda n-tartibli bir jinsli chiziqli differensial tenglamaning F.Y.S ning mavjudligi haqidagi teoremani isbotlagan edik. Lekin F.Y.S ni topish masalasi bilan shug'ullanmaganmiz.

Mazkur paragrafda, agar n-tartibli bir jinsli chiziqli differensial tenglama o'zgarmas koeffitsiyentli bo'lsa, u holda uning F.Y.S ni topish mumkinligini ko'rsatamiz. Ushbu

$$L[y] = y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} y' + a_n y = 0 \quad (2.10.1)$$

differensial tenglamani qaraymiz. Bu yerda $a_j = \text{const}$, $j=1, n$ haqiqiy o'zgarmas sonlar. O'zgarmas koeffitsiyentli (2.10.1) ko'rinishdagi differensial tenglamaning muhimligi shundaki, uning F.Y.S ni topish masalasi n-darajali algebraik tenglamaning ildizlarini o'rganish masalasiga keltiriladi.

Avvalo (2.10.1) differensial tenglamaning biror xususiy yechimini

$$y(x) = e^{\lambda x}, \quad \lambda = \text{const} \quad (2.10.2)$$

ko'rinishda izlaymiz. Bu funktsiyani ketma-ket n -marta differensiallab

$$y' = \lambda e^{\lambda x}, \quad y'' = \lambda^2 e^{\lambda x}, \dots, \quad e^{\lambda x}, \quad y^{(n)} = \lambda^n e^{\lambda x}$$

hosilalarni topamiz. So'ngra $L[y(x)]$ ni hisoblaymiz:

$$L[y] = L[e^{\lambda x}] = (\lambda^n + a_1 \lambda^{n-1} + \dots + a_{n-1}) e^{\lambda x} = M(\lambda) e^{\lambda x}. \quad (2.10.3)$$

Bunda, ushbu

$$M(\lambda) = \lambda^n + a_1 \lambda^{n-1} + \dots + a_{n-1} \lambda + a_n \quad (2.10.4)$$

n -darajali ko'phadga (2.10.1) differensial tenglamaning xarakteristik ko'phadi deyiladi.

Agar biror $\lambda = \lambda_0$ soni (2.10.4) xarakteristik ko'phadning ildizi, ya'ni

$$M(\lambda_0) = 0$$

bo'lsa, u holda $L[e^{\lambda_0 x}] = 0$ bo'lib, $y(x) = e^{\lambda_0 x}$ funktsiya (2.10.1) differensial tenglamaning xususiy yechimidan iborat bo'ladi. Bizga algebra kursidan ma'lumki,

$$M(\lambda) = 0 \quad (2.10.5)$$

xarakteristik tenglamani n ta ildizi mavjud.

1. Avvalo (2.10.5) xarakteristik tenglama n ta har xil oddiy ildizlarga ega bo'lgan holni ko'rib chiqamiz. Aniqlik uchun ushbu

$$\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n \quad (2.10.6)$$

sonlar (2.10.5) xarakteristik tenglamaning har xil $M(\lambda_j) = 0$, $j = \overline{1, n}$ oddiy ildizlari, ya'ni $M(\lambda_j) \neq 0$, $j = \overline{1, n}$ bo'lsin. U holda (2.10.6) ko'rinishdagi ildizlarga (2.10.1) differensial tenglamaning

$$y_1(x) = e^{\lambda_1 x}, \quad y_2(x) = e^{\lambda_2 x}, \dots, \quad y_n(x) = e^{\lambda_n x} \quad (2.10.7)$$

ko'rinishdagi xususiy yechimlari mos keladi. Bu yechimlar (2.10.1) differensial tenglamaning F.Y.S ni tashkil qilishini ko'rsatamiz. Shu maqsadda (2.10.8) yechimlardan tuzilgan Vronskiy determinantini tuzamiz va uning son qiymatini topamiz:

$$W \{y_1(x), y_2(x), \dots, y_n(x)\} =$$

$$\begin{aligned}
&= \begin{vmatrix} y_1(x) & y_2(x) & \dots & y_n(x) \\ y_1'(x) & y_2'(x) & \dots & y_n'(x) \\ \dots & \dots & \dots & \dots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \dots & y_n^{(n-1)}(x) \end{vmatrix} = \\
&= \begin{vmatrix} e^{\lambda_1 x} & e^{\lambda_2 x} & \dots & e^{\lambda_n x} \\ \lambda_1 e^{\lambda_1 x} & \lambda_2 e^{\lambda_2 x} & \dots & \lambda_n e^{\lambda_n x} \\ \dots & \dots & \dots & \dots \\ \lambda_1^{n-1} e^{\lambda_1 x} & \lambda_2^{n-1} e^{\lambda_2 x} & \dots & \lambda_n^{n-1} e^{\lambda_n x} \end{vmatrix} = \\
&= e^{(\lambda_1 + \lambda_2 + \dots + \lambda_n)x} \begin{vmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_n \\ \dots & \dots & \dots & \dots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \dots & \lambda_n^{n-1} \end{vmatrix} = e^{(\lambda_1 + \lambda_2 + \dots + \lambda_n)x} \prod_{i \neq j} (\lambda_i - \lambda_j) \neq 0
\end{aligned}$$

Algebra kursida oxirgi determinantga Vandermond determinanti deyiladi.

U noldan farqli bo'lishi uchun λ_j larning har xil bo'lishi zarur va yetarli.

Shunday qilib, agar λ_j lar har xil bo'lsa, u holda (1) differensial tenglamaning

$$y_1(x) = e^{\lambda_1 x}, y_2(x) = e^{\lambda_2 x}, y_n(x) = e^{\lambda_n x}$$

yechimlari chiziqli bog'lanmagan bo'ladi. Shuning uchun ular (2.10.1) differensial tenglamaning F.Y.S ni tashkil qiladi. Demak, (2.10.1) differensial tenglamaning umumiy yechimi

$$y(x) = \sum_{j=1}^n c_j y_j(x) = \sum_{j=1}^n c_j e^{\lambda_j x} \quad (2.10.8)$$

ko'rinishda ifodalanadi. Bu yerda $c_j = \text{const}$, $j = \overline{1, n}$ - ixtoriy o'zgarmas sonlar.

Faraz qilaylik, (2.10.5) xarakteristik tenglamaning $\lambda = \lambda_j$ ildizlari orasida kompleks ildizlar ham bo'lsin. Masalan, $\lambda = \lambda_1 = \alpha + i\beta$, $\beta \neq 0$, $i = \sqrt{-1}$, $\lambda_2 = \alpha - i\beta$ bu ildizlarga ushbu

$$y_1(x) = e^{(\alpha + i\beta)x} = u_1 + iu_2,$$

$$y_2(x) = e^{(\alpha - i\beta)x} = u_1 - iu_2,$$

ko'rinishida kompleks yechimlar mos keladi. Quyidagi

$$u_1(x) = \frac{y_1(x) + y_2(x)}{2}, u_2(x) = \frac{y_1(x) - y_2(x)}{2i}$$

tengliklar o'rinli bo'lgani uchun, ushbu

$$u_1(x) = e^{\alpha x} \cos \beta x, \quad u_2(x) = e^{\alpha x} \sin \beta x, \quad (2.10.9)$$

$u_1(x)$ va $u_2(x)$ funksiyalar ham (2.10.1) differensial tenglamaning yechimlaridan iborat bo‘ladi.

Endi $\{y_1(x), y_2(x), \dots, y_n(x)\} = F.Y.Sda$ (2.10.5) xarakteristik tenglamaning kompleks ildizlariga mos keluvchi har bir $y_1(x), y_2(x)$ - kompleks qo‘shma yechimlari juftliklarini $u_1(x), u_2(x)$ -haqiqiy yechimlari juftligi bilan almashtiramiz. Shu bilan bir qatorda $y_j(x) = e^{\lambda_j x}$ ko‘rinishidagi haqiqiy yechimlarini $u_j(x) = y_j(x)$ deb olamiz. Natijada (2.10.1) differensial tenglama $u_1(x), u_2(x) \dots u_n(x)$ - haqiqiy yechimlarga ega bo‘ladi. Bu yechimlarning ixtiyoriy $\forall (x_1, x_2) \subset R$ intervalda chiziqli erkli ekanligini ko‘rsatamiz.

Faraz qilaylik, biror b_1, b_2, \dots, b_n sonlar uchun ushbu

$$b_1 u_1(x) + b_2 u_2(x) + \dots + b_n u_n(x) = 0, \quad \forall (x_1, x_2) \subset R$$

tenglik o‘rinli bo‘lsin. Bu yerda $u_1(x), u_2(x) \dots u_n(x)$ larni $y_1(x), y_2(x), \dots, y_n(x)$ lar bilan almashtirib,

$$d_1 y_1(x) + d_2 y_2(x) + \dots + d_n y_n(x) \equiv 0$$

munosabatni hosil qilamiz. Bunda $d_1 = \frac{b_1 - ib_2}{2}$, $d_2 = \frac{b_1 + ib_2}{2}$ va xuddi shuningdek, $y_{2p-1}(x), y_{2p}(x)$ - kompleks qo‘shma juftlik uchun d_{2p-1}, d_{2p} ; haqiqiy $y_1(x)$ uchun esa $u_r = y_r, d_r = b_r$ deb olamiz. Agar birorta $b_j \neq 0$ bo‘lsa, u holda $d_k \neq 0$ topilib, $y_1(x), y_2(x), \dots, y_n(x)$ funksiyalar chiziqli bog‘liq bo‘ladi. Bu esa (2.10.5) xarakteristik tenglamani har xil ildizlariga mos keluvchi $y_1(x), y_2(x), \dots, y_n(x)$ yechimlarining chiziqli bog‘lanmaganligiga zid. Shuning uchun barcha $b_j = 0, j = \overline{1, n}$ bo‘lib, $u_1(x), u_2(x) \dots u_n(x)$ yechimlar chiziqli bog‘lanmagan bo‘ladi.

Shunday qilib, (2.10.5) xarakteristik tenglamaning oddiy λ_j ildizlariga haqiqiy funksiyalardan tashkil topgan F.Y.S mavjud ekan. Xarakteristik tenglamaning har bir haqiqiy λ_j ildizlariga $e^{\lambda_j x}$ ko‘rinishdagi funksiyalar va uning qo‘shma kompleks $\lambda = \alpha \pm i\beta, \beta \neq 0$ ildizlariga esa $e^{\lambda x} \cos \beta x, e^{\lambda x} \sin \beta x$ ko‘rinishdagi funksiyalar (2.10.1) differensial tenglamaning F.Y.S ni tashkil qiladi.

2. Karrali ildizlar holi. Avvalo $y(x) = x^p e^{\lambda x}$ ga nisbatan, (2.10.1) tenglamaning chap tomonidagi $L[y]$ ifodaning qiymatini hisoblaymiz. Bu yerda $p \geq 0$ - butun son.

2.10.1-teorema. Agar λ soni (2.10.5) xarakteristik tenglamaning k karrali ildizi bo‘lsa, u holda

$$L[x^p e^{\lambda x}] = \begin{cases} 0, & p \leq k-1 \\ (d_0 x^m + d_1 x^{m-1} + \dots + d_m) e^{\lambda x}, & p \geq k \end{cases} \quad (2.10.10)$$

munosabat o'rinli bo'ladi. Bu yerda $d_0 \neq 0$, $m = p - k$

Isbot. Teoremani isbotlashdan oldin quyidagi

$$L[y] = y''' + a_1 y'' + a_2 y' + a_3 y = 0 \quad (2.10.11)$$

xususiyl hol uchun $L[x^p e^{\lambda x}]$ ifodani hisoblaymiz. Buning uchun quyidagi

$$(x^p e^{\lambda x})' = e^{\lambda x} (px^{p-1} + \lambda x^p),$$

$$(x^p e^{\lambda x})'' = e^{\lambda x} (p(p-1)x^{p-2} + 2\lambda px^{p-1} + \lambda^2 x^p),$$

$$(x^p e^{\lambda x})''' = e^{\lambda x} (3\lambda p(p-1)x^{p-2} + 3\lambda^2 px^{p-1} + p(p-1)(p-2)x^{p-3} + \lambda^3 x^p)$$

sodda tengliklardan foydalanib, ushbu

$$L[x^p e^{\lambda x}] = e^{\lambda x} (3\lambda p(p-1)x^{p-2} + 3\lambda^2 px^{p-1} + p(p-1)(p-2)x^{p-3} + \lambda^3 x^p) +$$

$$+ a_1 e^{\lambda x} (p(p-1)x^{p-2} + 2\lambda px^{p-1} + \lambda^2 x^p) + a_2 e^{\lambda x} (px^{p-1} + \lambda x^p) + a_3 x^p e^{\lambda x} =$$

$$= e^{\lambda x} [M(\lambda)x^p + \frac{M'(\lambda)}{1!} px^{p-1} + \frac{M''(\lambda)}{2!} p(p-1)x^{p-2} +$$

$$+ \frac{M'''(\lambda)}{3!} p(p-1)(p-2)x^{p-3}],$$

ya'ni

$$L[x^p e^{\lambda x}] = e^{\lambda x} [M(\lambda)x^p + \frac{M'(\lambda)}{1!} px^{p-1} + \frac{M''(\lambda)}{2!} p(p-1)x^{p-2} + \frac{M'''(\lambda)}{3!} p(p-1)(p-2)x^{p-3}] \quad (2.10.12)$$

formulani topamiz.

1. Aytaylik $\lambda = \lambda_0$ soni, ushbu

$$M(\lambda) = \lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3$$

xarakteristik ko'phadning $k=2$ ikki karrali ildizi, ya'ni

$$M(\lambda_0) = 0, \quad M'(\lambda_0) = 0, \quad M''(\lambda_0) \neq 0$$

bo'lsin. Bu holda (2.10.12) tenglikdan

$$L[x^p e^{\lambda_0 x}] = \begin{cases} 0, & p \leq 1 \\ e^{\lambda_0 x} M''(\lambda_0), & p = 2 \\ e^{\lambda_0 x} \left[\frac{p(p-1)}{2!} M''(\lambda_0) x^{p-2} + \frac{M'''(\lambda_0)}{3!} p(p-1)(p-2) x^{p-3} \right], & p \geq 3 \end{cases}$$

munosabatga ega bo'lamiz. Demak, λ_0 ikki karrali ildizga (2.10.11) tenglamaning

$$e^{\lambda_0 x}, \quad x e^{\lambda_0 x}$$

xususiy yechimlari mos keladi.

2. Aytaylik, $\lambda = \lambda_0$ soni $M(\lambda_0) = 0$ xarakteristik tenglamaning uch karrali ildizi, ya'ni

$$M(\lambda_0) = 0, M'(\lambda_0) = 0, M''(\lambda_0) = 0, M'''(\lambda_0) \neq 0$$

bo'lsin. Qaralayotgan holda (2.10.12) formuladan foydalanib,

$$L[x^p e^{\lambda_0 x}] = \begin{cases} 0, & p \leq 2 \\ e^{\lambda_0 x} \left[\frac{M'''(\lambda_0)}{3!} p(p-1)(p-2)x^{p-3} \right], & p \geq 3 \end{cases}$$

tenglikni olamiz. Demak, uch karrali ildiz holiga (2.10.11) differensial tenglamaning

$$e^{\lambda_0 x}, x e^{\lambda_0 x}, x^2 e^{\lambda_0 x}$$

ko'rinishdagi xususiy yechimlariga ega bo'lamiz.

2.10.1-Lemma. Ushbu

$$\begin{aligned} L[f(x)e^{\lambda x}] &= (f(x)e^{\lambda x})^{(n)} + a_1(f(x)e^{\lambda x})^{(n-1)} + \dots + a_{n-1}(f(x)e^{\lambda x})' + \\ &+ a_n(f(x)e^{\lambda x}) = e^{\lambda x} \left[M(\lambda)f(x) + \frac{M'(\lambda)}{1!} f'(x) + \frac{M''(\lambda)}{2!} f''(x) + \dots + \right. \\ &\left. + \frac{M^{(n)}(\lambda)}{n!} f^{(n)}(x) \right] \end{aligned} \quad (2.10.13)$$

ayniyat o'rinli. Bu yerda $f(x) \in C^{(n)}(R)$.

Isbot. Ayniyatni isbotlash uchun Leybnisning ushbu

$$\begin{aligned} (uv)^{(n)} &= u^{(n)}v + C_1^n u^{(n-1)}v' + C_2^n u^{(n-2)}v'' + \dots + \\ &+ C_1^n v^{(n-1)}u' + uv^{(n)} \end{aligned}$$

formulasidan foydalanamiz:

$$e^{\lambda x} f(x) = e^{\lambda x} f(x),$$

$$(e^{\lambda x} f(x))' = \lambda e^{\lambda x} f(x) + e^{\lambda x} f'(x) = e^{\lambda x} (\lambda f(x) + f'(x)),$$

$$(e^{\lambda x} f(x))'' = \lambda^2 e^{\lambda x} f(x) + \lambda e^{\lambda x} f'(x) + \lambda e^{\lambda x} f'(x) + e^{\lambda x} f''(x) =$$

$$= e^{\lambda x} (\lambda^2 + 2\lambda f'(x) + f''(x)) = e^{\lambda x} \left(\lambda^2 f + \frac{(\lambda^2)'}{1!} f' + \frac{(\lambda^2)''}{2!} f'' \right),$$

.....

$$\begin{aligned} (e^{\lambda x} f(x))^{(n)} &= e^{\lambda x} \left(\lambda^n f(x) + \frac{(\lambda^n)'}{1!} f'(x) + \dots + \frac{(\lambda^n)^{(k)}}{k!} f^{(k)}(x) + \right. \\ &\left. + \dots + \frac{(\lambda^n)^{(n)}}{n!} f^{(n)}(x) \right). \end{aligned}$$

Bu tengliklarni mos ravishda a_j larga ko'paytirib, quyidagi

$$\begin{aligned}
a_n e^{\lambda x} f(x) &= a_n e^{\lambda x} f(x) \\
a_{n-1} (e^{\lambda x} f(x))' &= a_{n-1} e^{\lambda x} (\lambda f(x) + f'(x)) \\
a_{n-2} (e^{\lambda x} f(x))'' &= a_{n-2} e^{\lambda x} (\lambda^2 f + \frac{(\lambda^2)'}{1!} f' + \frac{(\lambda^2)''}{2!} f'') \\
&\cdot \\
&\cdot \\
&\cdot \\
(e^{\lambda x} f(x))^{(n)} &= e^{\lambda x} (\lambda^n f(x) + \frac{(\lambda^n)'}{1!} f'(x) + \dots + \frac{(\lambda^n)^{(k)}}{k!} f^{(k)}(x) + \\
&+ \dots + \frac{(\lambda^n)^{(n)}}{n!} f^{(n)}(x))
\end{aligned}$$

munosabatlarni hosil qilamiz. Oxirgi tengliklarni hadlab qo‘shish natijasida ushbu

$$\begin{aligned}
L[e^{\lambda x} f(x)] &= (e^{\lambda x} f(x))^{(n)} + a_1 (e^{\lambda x} f(x))^{(n-1)} + \dots + \\
&+ a_{n-1} (e^{\lambda x} f(x))' + a_n (e^{\lambda x} f(x)) = \\
&= e^{\lambda x} [(\lambda^n + a_1 \lambda^{n-1} + \dots + a_{n-1} \lambda + a_n) f(x) + \\
&\quad + \frac{(\lambda^n + a_1 \lambda^{n-1} + \dots + a_{n-1} \lambda + a_n)'}{1!} f'(x) + \dots + \\
&\quad + \frac{(\lambda^n + a_1 \lambda^{n-1} + \dots + a_{n-1} \lambda + a_n)^{(n)}}{n!} f^{(n)}(x)] = \\
&= e^{\lambda x} [M(\lambda) f(\lambda) + \frac{M'(\lambda)}{1!} f'(x) + \dots + \frac{M^{(k)}(\lambda)}{k!} f^{(k)}(x) + \\
&\quad + \dots + \frac{M^{(n)}(\lambda)}{n!} f^{(n)}(\lambda)]
\end{aligned}$$

formulani olamiz. ■

2.10.1-natija. Agar $f(x) = x^p$ ko‘rinishda bo‘lsa, u holda (2.10.13) ayniyat quyidagi ko‘rinishga keladi:

$$\begin{aligned}
L[x^p e^{\lambda x}] &= e^{\lambda x} [M(\lambda) x^p + \frac{M'(\lambda)}{1!} p x^{p-1} + \dots + \\
&\quad + \frac{M^{(k)}(\lambda)}{k!} p(p-1)\dots(p-k+1) x^{p-k} + \dots + M^{(p)}(\lambda)]
\end{aligned} \tag{2.10.14}$$

Agar $p = 0$ bo‘lsa u holda

$$L[e^{\lambda x}] = e^{\lambda x} M(\lambda) \tag{2.10.15}$$

tenglikka ega bo‘lamiz.

Faraz qilaylik, (2.10.18) ko‘rinishdagi yechimlar chiziqli bog‘langan bo‘lsin. U holda

$$\sum_{r=1}^k (A_0^{(r)} + A_1^{(r)}x + \dots + A_{m_r-1}^{(r)}x^{m_r-1})e^{\lambda_r x} = 0,$$

ya’ni

$$\sum_{r=1}^k P_r(x)e^{\lambda_r x} = 0 \quad (2.10.19)$$

tenglikni qaraymiz. Bu yerda $A_j^{(r)} = \text{const}$, $P_r(x)$ ko‘phadni kamida bitta koefitsiyenti noldan farqli. (2.10.19) tenglikni quydagi

$$P_1(x) + \sum_{r=2}^n P_r(x)e^{(\lambda_r - \lambda_1)x} = 0$$

ko‘rinishda yozib olamiz va uni m_1 marta differensiallab, ushbu

$$\sum_{r=2}^k Q_r(x)e^{(\lambda_r - \lambda_1)x} = 0 \quad (2.10.20)$$

munosabatni topamiz. Bunda $Q_r(x)$ nolga teng bo‘lmagan ko‘phad. Yuqoridagi jarayonni davom ettirsak, quyidagi

$$R_r(x)e^{(\lambda_r - \lambda_{r-1})x} = 0$$

tenglik hosil bo‘ladi. Bunday bo‘lishi mumkin emas. Chunki $e^{(\lambda_r - \lambda_{r-1})x} \neq 0$ $R_r(x)$ -esa noldan farqli koefitsiyentga ega bo‘lgan ko‘phad. Shuning uchun $R_r(x) \neq 0$, $x \notin (x_1, x_2) \subset R$. Bundan esa yuqoridagi n ta yechimlarning chiziqli bog‘lanmaganligi kelib chiqadi. Demak, (2.10.18) ko‘rinishdagi funksiyalar (2.10.1) differensial tenglamaning F.Y.S ni tashkil qilar ekan.

2.10.2-natija. Aytaylik, $M(\lambda) = 0$ xarakteristik tenglama k_1, k_2, \dots, k_m karrali $\lambda_1, \lambda_2, \dots, \lambda_m$ ($m \in \mathbb{N}, 1 \leq m \leq n$) ($k_1 + k_2 + \dots + k_m = n$) har xil ildizlarga ega bo‘lsin. U holda (2.10.1) differensial tenglamaning ixtiyoriy yechimi

$$y(x) = P_1(x)e^{\lambda_1 x} + P_2(x)e^{\lambda_2 x} + \dots + P_m(x)e^{\lambda_m x}$$

ko‘rinishda bo‘ladi. Bunday ko‘rinishdagi funksiya (2.10.1) differensial tenglamaning yechimidan iborat bo‘ladi.

Bu yerda

$$P_j(x) = C_0^j + C_1^j x + \dots + C_{k_j-1}^j x^{k_j-1}$$

$k_j - 1$ darajali ko‘phad bo‘lib, uning $C_1^j, C_2^j, \dots, C_{k_j-1}^j$ koefitsiyentlari ixtiyoriy o‘zgarmas sonlar. Yuqoridagi tasdiqni quyidagicha ham bayon qilish mumkin.

2.10.2-lemma. Agar ushbu

$$P_1(x)e^{\lambda_1 x} + P_2(x)e^{\lambda_2 x} + \dots + P_m(x)e^{\lambda_m x} = 0, \quad \forall x \in R \quad (2.10.21)$$

tenglik ixtiyoriy $\forall x \in R$ lar uchun bajarilsa, u holda barcha $P_1(x), P_2(x), \dots, P_m(x)$ ko'phadlarning koeffitsiyentlari nolga teng bo'ladi. Bu yerda $\lambda_1, \lambda_2, \dots, \lambda_m$ lar xarakteristik tenglamaning k_1, k_2, \dots, k_m karrali har xil ildizlari.

Isbot. Lemmani isbotlash jarayonida matematik induksiya usulidan foydalanamiz. $m=1$ holda 2.10.2-lemmani isboti ravshan.

Aytaylik, $m-1$ uchun 2.10.2-lemma o'rinli bo'lsin. $\forall m > 1$ uchun 2.10.2-lemmani isbotlaymiz. Buning uchun quyidagi

$$P_1(x) + \sum_{k=2}^m P_k(x)e^{(\lambda_k - \lambda_1)x} = 0$$

tenglikni qaraymiz. Bunda $P_1(x)$ -ko'phadning darajasi n . Shuning uchun bu tenglikni $(N+1)$ marta differensiallab

$$\sum_{k=2}^m [P_k(x)e^{(\lambda_k - \lambda_1)x}]^{(N+1)} = 0 \quad (2.10.22)$$

tenglikni topamiz. Chunki $P_1^{(N+1)}(x) = 0$. Yuqoridagi (2.10.22) tenglikni

$$\sum_{k=2}^m Q_k(x)e^{(\lambda_k - \lambda_2)x} = 0$$

ko'rinishda yozish mumkin. Bu yerda $Q_k(x)$ ko'phadning darajasi $P_k(x)$ ning darajasi bilan bir xil, chunki $\lambda_k - \lambda_1 \neq 0$, $k = \overline{2, m}$. Induksiya shartiga ko'ra $Q_k(x) \equiv 0, \forall k = \overline{2, m}$. Bundan esa $P_k(x) \equiv 0, \forall k = \overline{2, m}$ kelib chiqadi. U holda $P_1(x) \equiv 0$ bo'ladi. Bu esa (2.10.21) dagi $P_1(x), P_2(x), \dots, P_m(x)$ ko'phadlarning barcha koeffitsiyentlari noldan iborat ekanligini ko'rsatadi. Shunday qilib, (2.10.1) differensial tenglamaning umumiy yechimini

$$y(x) = \sum_{r=1}^k Q_r(x)e^{\lambda_r x} \quad (2.10.23)$$

ko'rinishda yozish mumkin. Bu yerda $Q_r(x), m_r - 1$ darajali ko'phad bo'lib, uning koeffitsiyenti ixtiyoriy o'zgarmas sonlardan iborat. (2.10.21) tenglikdagi o'zgarmaslarning soni $m_1 + m_2 + \dots + m_k = n$ tenglikni qanoatlantiradi.

2. Agar $M(\lambda) = \lambda^n + a_1 \lambda^{n-1} + \dots + a_{n-1} \lambda + a_n = 0$ xarakteristik tenglama r karrali $\lambda = \alpha + i\beta, \beta \neq 0$ ko'rinishdagi kompleks ildizga ega bo'lsa, u holda bu ildizga (2.10.1) differensial tenglamaning

$$e^{(\alpha+i\beta)x}, xe^{(\alpha+i\beta)x}, \dots, x^{r-1}e^{(\alpha+i\beta)x} \quad (2.10.24)$$

ko'rinishdagi yechimlari mos keladi. Eyler formulasiga ko'ra

$$e^{(\alpha+i\beta)x} = e^{\alpha x} (\cos \beta x + i \sin \beta x)$$

tenglikni yozish mumkin. (2.10.24) yechimlarning haqiqiy va mavhum qismlarini ajratib quyidagi $2r$ ta haqiqiy yechimlarini hosil qilamiz:

$$e^{\alpha x} \cos \beta x, x e^{\alpha x} \cos \beta x, \dots, x^{r-1} e^{\alpha x} \cos \beta x, \quad (2.10.25)$$

$$e^{\alpha x} \sin \beta x, x e^{\alpha x} \sin \beta x, \dots, x^{r-1} e^{\alpha x} \sin \beta x.$$

$M(\lambda) = 0$ xarakteristik tenglamaning r karrali qo'shma kompleks $\bar{\lambda} = \alpha - i\beta$ ildiziga ham (2.10.25) ko'rinishdagi chiziqli bog'lanmagan yechimlar mos keladi.

Shunday qilib, xarakteristik tenglamaning r karrali kompleks ildiziga (2.10.1) differensial tenglamaning (2.10.25) ko'rinishdagi $2r$ ta haqiqiy yechimlari mos keladi.

Mustaqil yechish uchun mashqlar [8], §15, №432-453; [21], §11, №511-531.

11-§. Eyler tenglamasi

Ushbu

$$x^n y^{(n)} + a_1 x^{n-1} y^{(n-1)} + \dots + a_{n-1} x y' + a_n y = 0 \quad (2.11.1)$$

ko'rinishdagi differensial tenglamaga Eyler tenglamasi deyiladi. Bu yerda $a_j = \text{const}, j = \overline{1, n}$. Bu differensial tenglamani

$$x = e^t \quad (2.11.2)$$

almashtirish yordamida n -tartibli bir jinsli o'zgarmas koeffitsientli differensial tenglamaga keltirish mumkin. Haqiqatan ham, (2.11.2) almashtirish (2.11.1) differensial tenglamaning tartibini va chiziqililigini saqlab qoladi. Quyidagi hisoblashlarni bajaramiz:

$$\frac{dx}{dt} = e^t, \frac{dt}{dx} = e^{-t}$$

$$y' = \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} e^{-t} = e^{-t} \frac{dy}{dt},$$

$$x y' = e^t e^{-t} \frac{dy}{dt} = \frac{dy}{dt}$$

$$y'' = \frac{d^2 y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dt} \left(\frac{dy}{dx} \right) \frac{dt}{dx} =$$

$$= e^{-t} \frac{d}{dt} \left(e^{-t} \frac{dy}{dt} \right) = e^{-2t} \left(\frac{d^2 y}{dt^2} - \frac{dy}{dt} \right),$$

$$x^2 y'' = e^{2t} e^{-2t} \left(\frac{d^2 y}{dt^2} - \frac{dy}{dt} \right) = \frac{d^2 y}{dt^2} - \frac{dy}{dt},$$

.....

$$y^{(k)} = \frac{d^k y}{dx^k} = e^{-kt} \left(\alpha_1 \frac{dy}{dt} + \alpha_2 \frac{d^2 y}{dt^2} + \dots + \alpha_k \frac{d^k y}{dt^k} \right)$$

$$\begin{aligned}
 x^k y^{(k)} &= e^{kt} e^{-kt} \left(\alpha_1 \frac{dy}{dt} + \alpha_2 \frac{d^2 y}{dt^2} + \dots + \alpha_k \frac{d^k y}{dt^k} \right) = \\
 &= \alpha_1 \frac{dy}{dt} + \alpha_2 \frac{d^2 y}{dt^2} + \dots + \alpha_k \frac{d^k y}{dt^k}.
 \end{aligned}
 \tag{2.11.3}$$

Bu yerda $\alpha_1, \alpha_2, \dots, \alpha_n$ - o'zgarmas sonlar. Topilgan (2.11.3) formulalardan foydalanib (2.11.1) differensial tenglamani quyidagi

$$\frac{d^n y}{dt^n} + b_1 \frac{d^{n-1} y}{dt^{n-1}} + \dots + b_{n-1} \frac{dy}{dt} + b_n y = 0
 \tag{2.11.4}$$

ko'rinishda yozish mumkin. Bunda b_i - o'zgarmas sonlar. Oxirgi (2.11.4) differensial tenglamaning xususiy yechimlari $y(t) = e^{kt}$ almashtirish yordamida topiladi.

Berilgan (2.11.1) differensial tenglamaning xususiy yechimlarini ushbu

$$y(x) = x^k
 \tag{2.11.5}$$

almashtirish yordamida topish mumkin.

Quyidagi

$$y' = kx^{k-1},$$

$$y'' = k(k-1)x^{k-2},$$

.....

$$y^{(n)} = k(k-1)\dots(k-n+1)x^{k-n}$$

hosilalarni (2.11.1) differensial tenglamaga qo'yib

$$k(k-1)(k-2)\dots(k-n+1) + a_1 k(k-1)\dots(k-n+2) +
 \tag{2.11.6}$$

$$+ \dots + a_{n-1} k + a_n = 0$$

algebraik tenglamani hosil qilamiz. Bu algebraik tenglamaga (2.11.1) differensial tenglamaning xarakteristik tenglamasi deyiladi va u (2.11.4) differensial tenglamaning xarakteristik tenglamasi bilan mos tushadi.

1. Agar (2.11.6) xarakteristik tenglamaning ildizlari k_1, k_2, \dots, k_n haqiqiy va har xil bo'lsa, u holda (2.11.1) Eyler differensial tenglamasining umumiy yechimi

$$y(x) = C_1 x^{k_1} + C_2 x^{k_2} + \dots + C_n x^{k_n}
 \tag{2.11.7}$$

ko'rinishda bo'ladi.

2. Agar (2.11.6) xarakteristik tenglama k_1, k_2, \dots, k_n har xil ildizlarga ega bo'lib, ular r_1, r_2, \dots, r_n karrali ($r_1 + r_2 + \dots + r_n = n$) bo'lsa, u holda (2.11.1) Eyler differensial tenglamasining umumiy yechimi

$$y(x) = P_1(\ln x) x^{r_1} + x^{r_2} P_2(\ln x) + \dots + x^{r_j} P_j(\ln x)
 \tag{2.11.8}$$

ko'rinishda bo'ladi. Bu yerda $P_j(t), t = \ln x$ ixtiyoriy $r_j - 1$ darajali ko'phad.

3. Agar (2.11.6) xarakteristik tenglama $k = \alpha + i\beta$, $\beta \neq 0$ ko‘rinishdagi r karrali kompleks ildizga ega bo‘lsa, u holda (2.11.1) Eyler differensial tenglamasi uchun ushbu

$$x^\alpha \cos(\beta \ln x), x^\alpha (\ln x) \cos(\beta \ln x), \dots, x^\alpha (\ln x)^{r-1} \cos(\beta \ln x)$$

$$x^\alpha \sin(\beta \ln x), x^\alpha (\ln x) \sin(\beta \ln x), \dots, x^\alpha (\ln x)^{r-1} \sin(\beta \ln x)$$

funksiyalar xususiy yechim bo‘ladi.

Mustaqil yechish uchun mashqlar [8], §15, №618-627, 628-635; [21], §11, №589-600.

12-§. n-tartibli bir jinsli bo‘lmagan chiziqli differensial tenglama

Ushbu

$$L[y] = y^{(n)} + p_1(x)y^{(n-1)} + \dots + p_{n-1}(x)y' + p_n(x)y = f(x) \quad (2.12.1)$$

bir jinsli bo‘lmagan differensial tenglamani qaraylik. Shu bilan bir qatorda quyidagi

$$L[y] = y^{(n)} + p_1(x)y^{(n-1)} + \dots + p_{n-1}(x)y' + p_n(x)y = 0 \quad (2.12.2)$$

bir jinsli differensial tenglamani ham qaraymiz. Bu yerda

$$P_j(x), f(x) \in C[a, b], j = \overline{1, n}. \quad (2.12.3)$$

2.12.1-teorema. Agar $y_1(x), y_2(x), \dots, y_n(x)$ funksiyalar (2.12.2) differensial tenglamaning F.Y.S dan iborat bo‘lib, $\overline{y(x)}$ funksiya (2.12.1) differensial tenglamaning birorta xususiy yechimi bo‘lsa, u holda (2.12.1) differensial tenglamaning ixtiyoriy yechimi

$$y(x) = \overline{y(x)} + \sum_{j=1}^n C_j y_j(x) \quad (2.12.4)$$

ko‘rinishda bo‘ladi. Bunda $C_j = \text{const}$ ixtiyoriy o‘zgarmas sonlar.

Isbot. Ushbu

$$z(x) = y(x) - \overline{y(x)}$$

ayirmani qaraymiz. Teorema shartiga ko‘ra

$$L[y] = f(x), L[\overline{y}] = f(x)$$

munosabatlar o‘rinli. Bu tangliklardan foydalanib $L[z]$ ifodaning qiymatini hisoblaymiz:

$$L[z] = L[y(x) - \overline{y(x)}] = L[y] - L[\overline{y}] = f(x) - f(x) = 0.$$

Bundan o‘z navbatida $z(x)$ funksiya (2.12.2) differensial tenglamaning yechimi ekanligi kelib chiqadi. Shuning uchun $z(x)$ funksiya (2.12.2) bir jinsli differensial tenglamaning $\{y_1(x), y_2(x), \dots, y_n(x)\}$ F.Y.S orqali ifodalanadi:

$$z(x) = \sum_{j=1}^n C_j y_j(x), \quad C_j = \text{const}. \quad (2.12.6)$$

(2.12.5) va (2.12.6) tengliklardan ushbu

$$y(x) = \bar{y}(x) + \sum_{j=1}^n C_j y_j(x)$$

tasvir kelib chiqadi.

1. Xususiy yechimni topishning Koshi usuli

Endi, (2.12.1) bir jinsli bo'lmagan differensial tenglamaning $\bar{y}(x)$ -xususiy yechimini topish bilan shug'ullanamiz.

2.12.1-ta'rif. Quyidagi

$$1) L_x[K(x,t)] = 0, \quad (2.12.7)$$

$$2) K(x,t)|_{x=t} = K'_x(x,t)|_{x=t} = \dots = K_x^{(n-2)}(x,t)|_{x=t} = 0,$$

$$K_x^{n-1}(x,t)|_{x=t} = 1, t \in (a,b) \subseteq R \quad (2.12.8)$$

shartlarni qanoatlantiruvchi $K(x,t)$ funksiyaga (2.12.2) bir jinsli differensial tenglamaning Koshi funksiyasi deyiladi. Bu yerda

$$L_x[K(x,t)] \equiv \frac{d^n}{dx^n}(K(x,t)) + P_1(x) \frac{d^{n-1}}{dx^{n-1}}(K(x,t)) + \dots + P_{n-1}(x) \frac{d}{dx}(K(x,t)) + P_n(x)K(x,t).$$

Yechimning mavjudligi va yagonaligi haqidagi Koshi teoremasiga asosan (2.12.2) differensial tenglamaning Koshi funksiyasi mavjud va yagonadir.

Aytaylik, $K(x,t)$ (2.12.2) bir jinsli differensial tenglamaning Koshi funksiyasi bo'lsin. Agar $y_1(x), y_2(x), \dots, y_n(x)$ funksiyalar (2.12.2) differensial tenglamaning F.Y.Sini tashkil qilsa, u holda quyidagi

$$K(x,t) = C_1 y_1(x) + C_2 y_2(x) + \dots + C_n y_n(x) \quad (2.12.9)$$

tasvir o'rinli bo'ladi. Bu (2.12.9) funksiyani (2.12.8) boshlang'ich shartlarga qo'yib C_j - o'zgarmlarining qiymatini topamiz:

$$C_j = W^{-1}(t) \cdot W_{nj}(t), \quad j = \overline{1, n}. \quad (2.12.10)$$

Bu yerda $W(t)$ -Vronskiy determinanti, $W_{nj}(t)$ esa $y_j(t)$ elementning algebraik to'ldiruvchisi. Shunday qilib, (2.12.9) va (2.12.10) tengliklardan $K(x,t)$ Koshi funksiyasining aniq ko'rinishini topamiz:

$$K(x,t) = W^{-1} \cdot \sum_{j=1}^n W_{nj}(t) y_j(x). \quad (2.12.11)$$

2.12.2-teorema. Agar $K(x,t)$ funksiya (2.12.2) differensial tenglamaning Koshi funksiyasi bo'lsa, u holda ushbu

$$\bar{y}(x) = \int_{x_0}^x K(x,t)f(t)dt \quad (2.12.12)$$

funksiya (2.12.1) differensial tenglamaning

$$\bar{y}(x_0) = \bar{y}'(x_0) = \dots = \bar{y}^{(n-1)}(x_0) \quad (2.12.13)$$

boshlang'ich shartlarni qanoatlantiruvchi xususiy yechimi bo'ladi.

Isbot. Yuqoridagi (2.12.12) tenglikni ketma-ket n marta differensiallasak va (2.12.8) boshlang'ich shartlardan foydalansak quyidagi

$$\begin{aligned} \bar{y}'(x) &= K(x,x)f(x) + \int_{x_0}^x K'_x(x,t)f(t)dt = \int_{x_0}^x K'_x(x,t)f(t)dt, \\ \bar{y}''(x) &= K'_x(x,t)|_{t=x}f(x) + \int_{x_0}^x K''_{xx}(x,t)f(t)dt = \\ &= \int_{x_0}^x K''_{xx}(x,t)f(t)dt, \end{aligned} \quad (2.12.14)$$

.....

$$\begin{aligned} \bar{y}^{(j)}(x) &= \int_{x_0}^x \frac{\partial^j K(x,t)}{\partial x^j} f(t)dt, \quad j = \overline{1, n-1}, \\ \bar{y}^{(n)}(x) &= \int_{x_0}^x \frac{\partial^n K(x,t)}{\partial x^n} f(t)dt + f(x) \end{aligned}$$

munosabatlarga ega bo'lamiz. Bu tasvirlardan $\bar{y}(x)$ funksiya (2.12.13) boshlang'ich shartlarni qanoatlantirishi kelib chiqadi. Endi (2.12.12) va (2.12.14) tasvirlardan foydalanib $\bar{y}(x)$ funksiya (2.12.1) differensial tenglamaning xususiy yechimi ekanligini ko'rsatamiz:

$$\begin{aligned} L[\bar{y}] &= \bar{y}^{(n)} + P_1(x)\bar{y}^{(n-1)} + \dots + P_{n-1}(x)\bar{y}' + P_n(x)y = \\ &= f(x) + \int_{x_0}^x \frac{\partial^n K(x,t)}{\partial x^n} f(t)dt + P_1(x) \int_{x_0}^x \frac{\partial^{n-1} K(x,t)}{\partial x^{n-1}} f(t)dt + \dots + \\ &+ P_{n-1}(x) \int_{x_0}^x \frac{\partial K(x,t)}{\partial x} f(t)dt + P_n(x) \int_{x_0}^x K(x,t)f(t)dt = f(x) + \\ &+ \int_{x_0}^x \left[\frac{\partial^n K(x,t)}{\partial x^n} + P_1(x) \frac{\partial^{n-1} K(x,t)}{\partial x^{n-1}} + \dots + P_{n-1}(x) \frac{\partial K(x,t)}{\partial x} + P_n(x)K(x,t) \right] \cdot \end{aligned}$$

$$\int_{x_0}^x f(t)dt = f(x) + \int_{x_0}^x L_x[K(x,t)]f(t)dt = f(x).$$

Shunday qilib, (2.12.1) differensial tenglamaning umumiy yechimi uchun quyidagi

$$y(x) = \sum_{j=1}^n C_j y_j(x) + \int_{x_0}^x \left[W^{-1}(t) \sum_{j=1}^n W_{nj}(t) y_j(x) \right] f(t) dt \quad (2.12.15)$$

Koshi formulasi o‘rinli bo‘lar ekan.

2. n-tartibli bir jinsli bo‘lmagan chiziqli differensial tenglamaning yechimini topishning Lagranj usuli

Agar (2.12.2) bir jinsli differensial tenglamaning $y_1(x), y_2(x), \dots, y_n(x)$ F.Y.S ma’lum bo‘lsa, u holda (2.12.1) bir jinsli bo‘lmagan differensial tenglamaning umumiy yechimini Lagranj (o‘zgarmasni variyatsiyalash) usulidan foydalanib topish mumkin. Buning uchun (2.12.1) bir jinsli bo‘lmagan differensial tenglamaning umumiy yechimini ushbu

$$y(x) = \sum_{j=1}^n C_j(x) y_j(x) \quad (2.12.16)$$

ko‘rinishda izlaymiz. Bu yerda $C_j(x)$ - hozircha noma’lum funksiyalar. Endi, $C_j(x)$ funksiyalarni shunday tanlaymizki, natijada quyidagi

$$y^{(j)}(x) = C_1(x) y_1^{(j)}(x) + C_2(x) y_2^{(j)}(x) + \dots + C_n(x) y_n^{(j)}(x), \quad j = \overline{1, n-1} \quad (2.12.16')$$

$$C_1'(x) y_1^{(j)}(x) + C_2'(x) y_2^{(j)}(x) + \dots + C_n'(x) y_n^{(j)}(x) = 0, \quad j = \overline{0, n-2}. \quad (2.12.17)$$

munosabatlar bajarilsin. Ushbu $y^{(j)}(x), j = \overline{0, n}$ hosilalarning (2.12.16') ifodalarini (2.12.1) differensial tenglamaga qo‘yib, $C_j'(x), j = \overline{1, n}$ larga nisbatan, yana bir tenglamani olamiz:

$$C_1'(x) y_1^{(n-1)}(x) + C_2'(x) y_2^{(n-1)}(x) + \dots + C_n'(x) y_n^{(n-1)}(x) + C_1(x) L[y_1] + C_2(x) L[y_2] + \dots + C_n(x) L[y_n] = f(x). \quad (2.12.18)$$

Bunda $y_1(x), y_2(x), \dots, y_n(x)$ funksiyalar (2.12.2) differensial tenglamaning yechimlari bo‘lgani uchun $L[y_1] = 0, L[y_2] = 0, \dots, L[y_n] = 0$ munosabatlar o‘rinli bo‘ladi. Shuning uchun (2.12.18) tenglik quyidagi ko‘rinishga keladi:

$$C_1'(x) y_1^{(n-1)}(x) + C_2'(x) y_2^{(n-1)}(x) + \dots + C_n'(x) y_n^{(n-1)}(x) = f(x). \quad (2.12.19)$$

Hosil bo‘lgan (2.12.17), (2.12.19) algebraik tenglamalar sistemasidan $C_j'(x), j = \overline{1, n}$ larni topish mumkin. Bu sistemaning asosiy determinanti $y_1(x), y_2(x), \dots, y_n(x)$ F.Y.S dan tuzilgan vronskiyan $W(x) = W\{y_1(x), y_2(x), \dots, y_n(x)\} \neq 0$ bilan bir xil. Kramer qoidasiga asosan quyidagi

$$C'_1(x) = W^{-1}(x)W_{n1}(x)f(x),$$

$$C'_2(x) = W^{-1}(x)W_{n2}(x)f(x),$$

.....

$$C'_n(x) = W^{-1}(x)W_{nn}(x)f(x)$$

tengliklarga ega bo‘lamiz. $W^{-1}(x), f(x)$ va $W_{nj}(x), j = \overline{1, n}$ funksiyalarning uzluksizligidan

$$C_j(x) = d_j + \int W^{-1}(x)W_{nj}(x)f(x)dx, \quad j = \overline{1, n} \quad (2.12.20)$$

formulaga ega bo‘lamiz. Bu yerdagi integral ushbu, $W^{-1}(x)W_{nj}(x)f(x)$ funksiyaning boshlang‘ichi, ya‘ni

$$\left(\int W^{-1}(x)W_{nj}(x)f(x) \right)' = W^{-1}(x)W_{nj}(x)f(x),$$

$d_j, j = \overline{1, n}$ -ixtiyoriy o‘zgarmas sonlar. (2.12.20) tenglik orqali topilgan $C_j(x), j = \overline{1, n}$ larni (2.12.16) formulaga qo‘yib, (2.12.1) differensial tenglamaning umumiy yechimini topamiz:

$$y(x) = \sum_{j=1}^n d_j y_j(x) + \int W^{-1}(x) \sum_{j=1}^n W_{nj}(x) f(x) dx. \quad (2.12.21)$$

Mustaqil yechish uchun mashqlar [8], §15, №653-666, 628-635; [21], §11, №575-580.

13-§. Ba‘zi o‘zgarmas koeffitsiyentli chiziqli bir jinsli bo‘lmagan differensial tenglamalar

I. Bir jinsli bo‘lmagan ushbu

$$L[y] = y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} y' + a_n y = P_m(x) e^{\mu x} \quad (2.13.1)$$

ko‘rinishdagi chiziqli differensial tenglama berilgan bo‘lsin. Bu yerda

$$P_m(x) = P_m x^m + P_{m-1} x^{m-1} + \dots + P_1 x + P_0, \quad m \geq 0, \quad (2.13.2)$$

$a_j = \text{const}, j = \overline{1, n}; P_j = \text{const}, j = \overline{0, m}; \mu = \text{const}$ berilgan sonlar.

1-hol. Aytaylik, μ soni ushbu

$$M(\lambda) = \lambda^n + a_1 \lambda^{n-1} + \dots + a_{n-1} \lambda + a_n \quad (2.13.3)$$

xarakteristik tenglamaning ildizi bo‘lmasin, ya‘ni $M(\mu) \neq 0$ bo‘lsin. Bu holda (2.13.1) differensial tenglamaning xususiy yechimini

$$y(x) = Q_m(x) e^{\mu x} \quad (2.13.4)$$

ko‘rinishda izlaymiz. Bunda

$$Q_m = q_m x^m + q_{m-1} x^{m-1} + \dots + q_1 x + q_0 \quad (2.13.5)$$

q_0, q_1, \dots, q_m -hozircha noma'lum sonlar.

Shartga ko'ra, $M(\mu) \neq 0$ bo'lgani uchun $q_j, j = \overline{0, m}$ sonlarni shunday tanlaymizki, natijada quyidagi

$$L[Q_m(x)e^{\mu x}] = P_m(x)e^{\mu x},$$

ya'ni

$$e^{-\mu x} L[Q_m(x)e^{\mu x}] = P_m(x) \quad (2.13.6)$$

munosabat bajarilsin.

Oldingi paragrafdagi (2.10.14), ya'ni ushbu

$$L[x^p e^{\mu x}] = e^{\mu x} \left[M(\lambda)x^p + \frac{M'(\lambda)}{1!} px^{p-1} + \dots + \frac{M^{(k)}(\lambda)}{k!} p(p-1)\dots(p-k+1)x^{p-k} + \dots + M^{(p)}(\lambda) \right] \quad (2.13.7)$$

formulaga asosan (2.13.6) tenglikning chap tomonini hisoblaymiz:

$$\begin{aligned} e^{-\mu x} L[Q_m e^{\mu x}] &= e^{-\mu x} L[(q_m x^m + \dots + q_0) e^{\mu x}] = \\ &= e^{-\mu x} q_m L[x^m e^{\mu x}] + e^{-\mu x} q_{m-1} L[x^{m-1} e^{\mu x}] + \dots + \\ &+ e^{-\mu x} q_0 L[e^{\mu x}] = q_m \left\{ M(\mu)x^m + \frac{M'(\mu)}{1!} mx^{m-1} + \dots + \right. \\ &+ \left. \frac{M^{(k)}(\mu)}{k!} m(m-1)\dots(m-k+1)x^{m-k} + \dots + M^{(m)}(\mu) \right\} + \\ &+ q_{m-1} \left\{ M(\mu)x^{m-1} + \frac{M'(\mu)}{1!} (m-1)x^{m-2} + \dots + M^{(m-1)}(\mu) \right\} + \\ &+ q_1 \{ M(\mu)x + M'(\mu) \} + q_0 M(\mu). \end{aligned}$$

Yuqoridagi (2.13.6) tenglikka ko'ra, ushbu

$$\begin{aligned} q_m M(\mu)x^m + \left[\frac{M'(\mu)}{1!} mq_m + q_{m-1} M(\mu) \right] x^{m-1} + \dots + \\ + \{ q_m M(\mu) + q_{m-1} M^{(m-1)}(\mu) + \dots + q_1 M'(\mu) + q_0 M(\mu) \} = \\ = P_m x^m + P_{m-1} x^{m-1} + \dots + P_0 \end{aligned}$$

munosabatga ega bo'lamiz. Bu ko'phadlarning tengligidan foydalanib, q_j - noma'lumlarga nisbatan, quyidagi tenglamalar sistemasini hosil qilamiz:

$$\begin{cases} q_m M(\mu) = P_m \\ q_{m-1} M(\mu) + q_m M'(\mu)m = P_{m-1}, \\ \dots \\ q_0 M(\mu) + q_1 M'(\mu) + \dots + q_{m-1} M^{(m-1)}(\mu) + q_m M^{(m)}(\mu) = P_0. \end{cases} \quad (2.13.8)$$

Bu sistemaning asosiy determinantini hisoblaymiz:

$$\Delta = \begin{vmatrix} 0 & 0 & \dots & 0 & M(\mu) \\ 0 & 0 & \dots & M(\mu) & mM'(\mu) \\ \dots & \dots & \dots & \dots & \dots \\ 0 & M(\mu) & \dots & M^{(m-2)}(\mu) & M^{(m-1)}(\mu) \\ M(\mu) & M'(\mu) & \dots & M^{(m-1)}(\mu) & M^{(m)}(\mu) \end{vmatrix} = [M(\mu)]^{m+1} \neq 0.$$

Shuning uchun bu tenglamalar sistemasining yagona yechimi mavjud. Endi (2.13.8) sistemani yechib q_m larni aniqlaymiz:

$$q_m = \frac{P_m}{M(\mu)},$$

$$q_{m-1} = \frac{P_{m-1}}{M(\mu)} - \frac{mP_m}{[M(\mu)]^2} \cdot M'(\mu), \quad (2.13.9)$$

.....

Demak, qaralayotgan holda ($M(\mu) \neq 0$) (2.13.4) ko‘rinishdagi yechimning tarkibidagi barcha q_j noma'lumlar (2.13.9) formulalar orqali yagona aniqlanadi.

2-hol. Aytaylik, $\lambda = \mu$ soni (2.13.3) xarakteristik tenglamaning r karrali ildizi, ya'ni

$$M(\mu) = 0, M'(\mu) = 0, \dots, M^{(r-1)}(\mu) = 0, M^{(r)}(\mu) \neq 0. \quad (2.13.10)$$

o‘rinli bo‘lsin. U holda, (2.13.7) formula ushbu

$$L[x^m e^{\mu x}] = e^{\mu x} \left[\frac{M^{(r)}(\mu)}{r!} m(m-1)\dots(m-k+1)x^{m-r} + \dots + M^{(m)}(\mu) \right] \quad (2.13.11)$$

ko‘rinishni oladi. Bu tenglikning o‘ng tomoni $(m-r)$ -darajali ko‘phaddan iborat. Shuning uchun (2.13.1) differensial tenglamaning xususiy yechimini

$$y(x) = x^r Q_m(x) e^{\mu x} = e^{\mu x} [q_m x^{m+r} + q_{m-1} x^{m+r-1} + \dots + q_0 x^r] \quad (2.13.12)$$

ko‘rinishda izlaymiz. (2.13.12) tenglik orqali aniqlangan $y(x)$ funksiyani (2.13.1) differensial tenglamaga qo‘yib $q_j, j = \overline{0, m}$ noma'lumlarni shunday tanlaymizki, natijada quyidagi

$$e^{-\mu x} L[x^r Q_m(x) e^{\mu x}] = P_m(x) \quad (2.13.13)$$

munosabat o‘rinli bo‘lsin. Bu tenglikni chap tomonini (2.13.10) va (2.13.11) munosabatdan foydalanib hisoblash mumkin:

$$e^{-\mu x} L[x^r Q_m(x) e^{\mu x}] = q_m \{ C_r^{m+r} x^m M^{(r)}(\mu) + C_{r+1}^{m+r} x^{m-1} M^{(r+1)}(\mu) + \dots + M^{(r+m)}(\mu) \} + q_{m-1} \{ C_r^{m+r-1} x^{m-1} M^{(r)}(\mu) + C_{r+1}^{m+r-1} x^{m-2} M^{(r+1)}(\mu) + \dots + M^{(r+m-1)}(\mu) \} + \dots + q_1 \{ C_r^{r+1} x M^{(r)}(\mu) + M^{(r+1)}(\mu) \} + q_0 M^{(r)}(\mu).$$

(2.13.14) va (2.13.13) tengliklarning o‘ng tomonlarini tenglashtirib, q_0, q_1, \dots, q_m noma'lumlarga nisbatan quyidagi algebraik tenglamalar sistemasini keltirib chiqaramiz:

ko‘rinishdagi tenglamaga ikkinchi tartibli chiziqli bir jinsli differensial tenglama deyiladi. Bu yerda $p_1(x) \in C^1(I)$ bir marta uzluksiz differensiallanuvchi, $p_2(x) \in C(I)$ uzluksiz funksiyalar. Bunda $I \subseteq R$ oraliq.

Quyidagi

$$y(x) = u(x)v(x) \quad (2.14.2)$$

almashtirish natijasida (2.14.1) differensial tenglamani ushbu

$$v''(x) + q(x)v(x) = 0 \quad (2.14.3)$$

ko‘rinishdagi ikki hadli tenglamaga keltirish mumkin. Bunda $q(x) \in C(I)$ uzluksiz funksiya. Haqiqatdan ham (2.14.2) tenglikdan

$$y' = u'v + uv', \quad y'' = u''v + 2u'v' + uv''$$

hosilalarni topib (2.14.1) tenglamaga qo‘ysak,

$$uv'' + (2u' + p_1(x)u)v' + (u'' + p_1(x)u' + p_2(x)u)v = 0 \quad (2.14.4)$$

hosil bo‘ladi. Bunda $u(x)$ funksiyani shunday tanlaymizki, natijada ushbu

$$2u' + p_1(x)u = 0$$

munosabat o‘rinli bo‘lsin. Bundan

$$u(x) = e^{-\frac{1}{2} \int_{x_0}^x p_1(t) dt}, \quad x_0 \in I \quad (2.14.5)$$

funksiyani topib olamiz. $u(x)$ funksiyani bunday tanlash natijasida

$$u' = -\frac{1}{2} p_1(x)u, \quad u'' = -\frac{1}{2} (p_1'(x)u + p_1(x)u') = -\frac{1}{2} (p_1'(x) - \frac{1}{2} p_1^2(x))u$$

tengliklarga ega bo‘lamiz. Topilgan $u(x), u'(x), u''(x)$ larning ifodalarini (2.14.4) tenglamaga qo‘yib, uni $u(x) \neq 0$ ga bo‘lsak, (2.14.3) tenglama hosil bo‘ladi. Bunda

$$q(x) = p_2(x) - \frac{1}{4} p_1^2(x) - \frac{1}{2} p_1'(x), \quad x \in I.$$

Yuqoridagi (2.14.2) va (2.14.5) tengliklardan ko‘rinadiki, $u(x)$ va $v(x)$ funksiyalarning I oraliqdagi nollari bir xil, shuning uchun kelajakda (2.14.3) ko‘rinishdagi ikki hadli differensial tenglamani o‘rganamiz.

2.14.1-lemma. (2.14.3) differensial tenglamaning har bir nolmas $v(x)$ yechimining ixtiyoriy $x_0 \in I$ noli oddiy, ya’ni $v(x_0) = 0, v'(x_0) \neq 0$.

Isbot. Faraz qilaylik, $x_0 \in I, v(x)$ – nolmas yechimning karrali, ya’ni $v(x_0) = 0, v'(x_0) = 0$ noli bo‘lsin. U holda Koshining yagonalik teoremasiga ko‘ra $v(x) \equiv 0, x \in I$ bo‘lar edi. Bu ziddiyat farazimizning noto‘g‘riligini ko‘rsatadi.

2.14.2-lemma. (2.14.3) tenglamaning har bir nolmas $0 \neq y(x)$ yechimi ixtiyoriy chekli $[a, b] \subset I$ oraliqda cheksiz ko‘p nolga ega bo‘lmaydi.

Isbot. Aytaylik $I \supset [a, b]$ kesmada (2.14.3) tenglamaning noldan farqli $0 \neq y(x)$ yechimi cheksiz ko'p nolga ega, ya'ni $\exists \{x_k\}_{k=1}^{\infty}$, $x_k \in [a, b]$ topilib $y(x_k) = 0, \forall k \in \mathbb{N}$ bo'lsin. U holda $y(x)$ funksiyaning uzluksiz differensiallanuvchiligiga asosan

$$y(x_0) = \lim_{k \rightarrow \infty} y(x_k) = 0, \quad y'(x_0) = \lim_{x_k \rightarrow x_0} \frac{y(x_k) - y(x_0)}{x_k - x_0} = 0$$

munosabatlarga ega bo'lamiz. Koshining yagonalik teoremasidan $y(x) \equiv 0, x \in [a, b]$ kelib chiqadi. Bu esa ziddiyat.

2.14.1-natija. (2.14.3) tenglamaning ixtiyoriy nolmas yechimi har bir chekli $[a, b] \subset I$ oraliqda faqat cheklita nolga ega.

Endi (2.14.3) ko'rinishdagi ikkita tenglamaning noldan farqli yechimlarining nollarini taqqoslaymiz.

2.14.1-teorema (Shturm). Quyidagi ikkita

$$y'' + q(x)y = 0, \quad x \in I \quad (2.14.6)$$

$$y'' + \tilde{q}(x)y = 0, \quad (2.14.7)$$

differensial tenglama berilgan bo'lib, $q(x) < \tilde{q}(x), x \in I$ bo'lsin, hamda $\varphi(x)$ va $\tilde{\varphi}(x)$ funksiyalar mos ravishda (2.14.6) va (2.14.7) tenglamalarning noldan farqli ixtiyoriy yechimlari bo'lsin. U holda $\varphi(x)$ yechimning ixtiyoriy ikkita ketma – ket kelgan nollari orasida $\tilde{\varphi}(x)$ yechimning kamida bitta noli bo'ladi.

Isbot. Ushbu

$$y'' + q(x)y = 0,$$

$$y'' + \tilde{q}(x)y = 0$$

ayniyatlarni mos ravishda $\tilde{\varphi}(x)$ va $\varphi(x)$ funksiyalarga ko'paytirib, hosil bo'lgan tengliklarning birinchisidan ikkinchisini ayirsak

$$\tilde{\varphi}''\varphi - \varphi''\tilde{\varphi} + [q(x) - \tilde{q}(x)]\varphi\tilde{\varphi} = 0,$$

ya'ni,

$$\varphi''\tilde{\varphi} - \tilde{\varphi}''\varphi = [q(x) - \tilde{q}(x)]\varphi\tilde{\varphi}$$

kelib chiqadi. Oxirgi tenglikni quyidagicha yozish mumkin:

$$(\varphi'\tilde{\varphi} - \tilde{\varphi}'\varphi)' = [\tilde{q}(x) - q(x)]\varphi\tilde{\varphi}. \quad (2.14.8)$$

Avvalo x_1 va x_2 orqali $\varphi(x)$ funksiyaning ketma – ket kelgan ixtiyoriy ikkita nolini belgilaymiz, ya'ni $\varphi(x_1) = 0, \varphi(x_2) = 0$. So'ngra (2.14.8) ayniyatni $[x_1, x_2]$ oraliq bo'yicha integrallasak,

$$\varphi'(x_2)\tilde{\varphi}(x_2) - \varphi'(x_1)\tilde{\varphi}(x_1) = \int_{x_1}^{x_2} [\tilde{q}(x) - q(x)]\varphi(x)\tilde{\varphi}(x)dx \quad (2.14.9)$$

tenglik hosil bo'ladi.

Endi $\tilde{\varphi}(x)$ funksiya (x_1, x_2) intervalda ildizga ega emas deb faraz qilaylik. Qulaylik uchun (x_1, x_2) intervalda $\varphi(x) > 0$, $\tilde{\varphi}(x) > 0$ deb hisoblashimiz mumkin (agar (x_1, x_2) oraliqda $\tilde{\varphi}(x) < 0$ bo'lsa, u holda $[-\tilde{\varphi}(x)]$ funksiyani olish mumkin). Ko'rinib turibdiki (2.14.9) tenglikning o'ng tomoni musbat bo'ladi. $\varphi(x_1) = 0$, $\varphi(x_2) = 0$, $\varphi(x) > 0$, $x \in (x_1, x_2)$ bo'lgani uchun $\varphi'(x_1) \geq 0$, $\varphi'(x_2) \leq 0$ o'rinli bo'ladi. Haqiqatdan ham

$$\varphi'(x_1) = \lim_{x \rightarrow x_1 + 0} \frac{\varphi(x)}{x - x_1} \geq 0, \quad \varphi'(x_2) = \lim_{x \rightarrow x_2 - 0} \frac{\varphi(x)}{x - x_2} \leq 0.$$

Agar $\varphi'(x_1) = 0$ yoki $\varphi'(x_2) = 0$ bo'lsa, yagonalik teoremasiga ko'ra $\varphi(x) \equiv 0$ bo'ladi. Shuning uchun $\varphi'(x_1) > 0$ va $\varphi'(x_2) < 0$ munosabatlar o'rinli bo'ladi. Bu tengsizliklardan

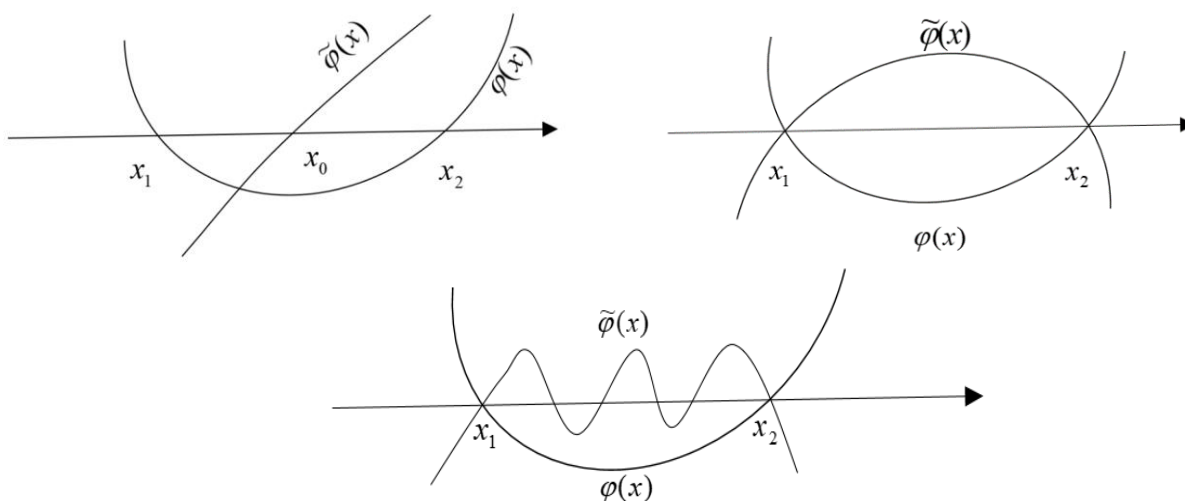
$$\varphi'(x_2)\tilde{\varphi}(x_2) - \varphi'(x_1)\tilde{\varphi}(x_1) < 0$$

baholash hosil bo'ladi. Bu tengsizlikdan esa ziddiyat kelib chiqadi, chunki (2.14.9) tenglikka asosan

$$\varphi'(x_2)\tilde{\varphi}(x_2) - \varphi'(x_1)\tilde{\varphi}(x_1) \geq 0$$

tengsizlik o'rinli bo'ladi. Bu esa ziddiyat.

Shturm teoremasining natijasini quyidagi chizmalarda izohlash mumkin:



1-chizma

Aytaylik, m va M sonlari ushbu $0 < m < M$ tengsizlikni qanoatlantirsin. U holda

$$y'' + m^2 y = 0$$

tenglama $y = \sin mx$ ko'rinishdagi, quyidagi

$$z'' + M^2 z = 0$$

tenglama esa $y = \sin Mx$ ko‘rinishdagi yechimga ega bo‘ladi. Bu $y = \sin mx$ va $y = \sin Mx$ yechimlar uchun Shturm teoremasining tasdiqlari o‘rinli bo‘ladi.

2.14.1-natija. Agar (2.14.3) tenglamada $q(x) \leq 0, \forall x \in I$ bo‘lsa, u holda uning ixtiyoriy nolmas yechimi I oraliqda ko‘pi bilan bitta nolga ega bo‘lishi mumkin.

Haqiqatdan ham, agar $y(x) \neq 0$ yechim I oraliqdagi x_1 va x_2 nuqtalarda $y(x_1) = 0, y(x_2) = 0$ nolga aylansa, u holda ushbu

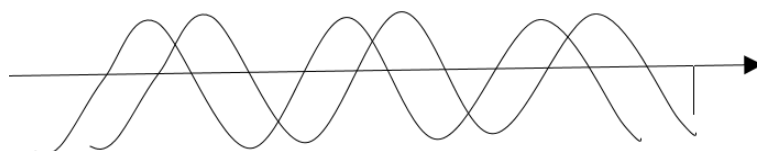
$$z''(x) + 0 \cdot z(x) = 0$$

tenglamaning $z(x)$ yechimi (x_1, x_2) oraliqda yotuvchi nuqtada nolga aylanar edi. Ammo $z(x) = 1$ yechim nolga aylanmaydi.

2.14.2-natija. Aytaylik, $y_1(x), y_2(x)$ funksiyalar (2.14.6) tenglamaning chiziqli erkli yechimlari bo‘lsin. Agar x_1, x_2 lar $y_1(x)$ -yechimning ketma-ket kelgan ikkita noli bo‘lsa, u holda $y_2(x)$ yechim (x_1, x_2) oraliqda faqat bitta nolga ega bo‘ladi. Boshqacha aytganda $y_1(x)$ va $y_2(x)$ funksiyalarning nollari almashinib keladi.

Isbot. Shturm teoremasiga ko‘ra (bu yerda $q(x) = \tilde{q}(x)$), $y_2(x)$ yechim $[x_1, x_2]$ oraliqda kamida bitta x_0 nuqtada nolga aylanadi. $y_1(x)$ va $y_2(x)$ funksiyalar umumiy nolga ega bo‘lmaydi. Agar, masalan $y_1(x_1) = 0, y_2(x_1) = 0$ bo‘lsa, u holda $w(x_1) = w\{y_1(x_1), y_2(x_1)\} = 0$ bo‘ladi. Bundan $y_1(x)$ va $y_2(x)$ yechimlarning chiziqli bog‘liqligi kelib chiqadi. Shuning uchun, $x_0 \in (x_1, x_2)$. Endi x_0 nuqta $y_2(x)$ funksiyaning (x_1, x_2) intervaldagi yagona noli ekanligini ko‘rsatamiz. Teskarisini faraz qilaylik, ya’ni $\exists \bar{x} \neq x_0, \bar{x} \in (x_1, x_2)$ nuqta topilib, $y(\bar{x}) = 0$ bo‘lsin. U holda Shturm teoremasiga ko‘ra $[x_0, \bar{x}]$ (aniqlik uchun $\bar{x} > x_0$ deylik) oraliqda $y_1(x)$ funksiya kamida bitta nolga ega bo‘ladi. Bu esa $y_1(x)$ funksiyaning x_1 va x_2 nollarining ketma-ket kelganligiga zid.

Bu natijani chizmada quyidagicha tasvirlash mumkin:



2-chizma

2.14.3-natija. Agar (2.14.6) tenglamaning biror noldan farqli yechimi I oraliqda cheksiz ko‘p nolga ega bo‘lsa, u holda uning ixtiyoriy noldan farqli yechimi shu oraliqda cheksiz ko‘p nolga ega bo‘ladi.

Ushbu $y'' + y = 0$ tenglamaning $y = \sin x$ va $y = \cos x$ chiziqli erkli yechimlari 2.14.2-natija va 2.14.3-natijalarga misol bo'la oladi.

2.14.1-misol. Faraz qilaylik, ushbu

$$y'' + q(x)y = 0$$

differential tenglamaning $q(x)$ - koeffitsenti quyidagi

$$0 < m \leq q(x) \leq M < +\infty, \quad m < M, \quad \forall x \in I$$

shartni qanoatlantirsin. U holda bu tenglamaning ixtiyoriy nolmas yechimining ketma – ket kelgan ikki noli orasidagi d - masofa

$$\frac{\pi}{\sqrt{M}} \leq d \leq \frac{\pi}{\sqrt{m}}$$

bahoni qanoatlantirishini ko'rsating.

Yechish. Avvalo ushbu

$$z'' + mz = 0$$

tenglamaning umumiy yechimini topamiz:

$$z(x) = A \sin(\sqrt{m}x + \varphi).$$

Agar x_1 va x_2 lar orqali $z(x)$ funksiyaning ketma – ket kelgan nollarini belgilasak, u holda

$$x_2 - x_1 = \frac{\pi}{\sqrt{m}}$$

tenglikka ega bo'lamiz. Haqiqatdan ham

$$z(x_1) = 0, \quad \sin(\sqrt{m}x_1 + \varphi) = 0, \quad \sqrt{m}x_1 + \varphi = n\pi,$$

$$z(x_2) = 0, \quad \sin(\sqrt{m}x_2 + \varphi) = 0, \quad \sqrt{m}x_2 + \varphi = (n+1)\pi,$$

$$\sqrt{m}x_2 - \sqrt{m}x_1 = \pi, \quad x_2 - x_1 = \frac{\pi}{\sqrt{m}}.$$

Endi \bar{x}_1 va \bar{x}_2 orqali $y'' + q(x)y = 0$ tenglamaning $y(x)$ yechimini $[x_1, x_2]$ oraliqdagi ketma – ket kelgan ikki nolini belgilaymiz. U holda Shturm teoremasiga ko'ra

$$x_1 \leq \bar{x}_1 < \bar{x}_2 \leq x_2$$

munosabat o'rinli bo'ladi. Bunda ushbu

$$\bar{x}_2 \leq x_2 = x_1 + \frac{\pi}{\sqrt{m}} \leq \bar{x}_1 + \frac{\pi}{\sqrt{m}},$$

ya'ni,

$$\bar{x}_2 - \bar{x}_1 \leq \frac{\pi}{\sqrt{m}}$$

bahoning o'rinli bo'lishi kelib chiqadi.

Endi ushbu

$$z'' + Mz = 0$$

tenglama uchun yuqoridagi mulohazani qo'llasak, ushbu

$$\bar{x}_2 - \bar{x}_1 \geq \frac{\pi}{\sqrt{M}}$$

bahoga ega bo'lamiz.

15-§. Ikkinchi tartibli chiziqli differensial tenglamaga qo'yilgan chegaraviy masalalar

Differensial tenglamaga qo'yilgan Koshi masalasi, shu tenglamaning berilgan nuqtadan o'tuvchi integral chizig'ini topishdan iborat edi. Klassik fizikaning va tatbiqiy matematikaning bir qator masalalari differensial tenglamaning berilgan ikki nuqtadan o'tuvchi integral chizig'ini topish masalasiga keltiriladi. Bu masala Koshi masalasidan tubdan farq qiladi. Chunki berilgan ikki nuqtaning har biri uchun alohida qo'yilgan Koshi masalalari yechimga ega bo'lsa ham, yuqoridagi masala yechimga ega bo'lishi yoki bo'lmasligi ham mumkin. Odatda bunday turdagi masalaga differensial tenglama uchun ikki nuqtali chegaraviy masala deb ataladi.

Bundan buyon ikkinchi tartibli chiziqli differensial tenglamaga

$$y'' + p_1(x)y' + p_2(x)y = f(x), \quad x \in [0, l] \quad (2.15.1)$$

qo'yilgan ushbu

$$l_1(y) \equiv \alpha_1 y(0) + \beta_1 y'(0) = y_0, \quad (2.15.2)$$

$$l_2(y) \equiv \alpha_2 y(l) + \beta_2 y'(l) = y_1$$

ko'rinishdagi chegaraviy masalani qaraymiz. Bu yerda $p_1(x), p_2(x), f(x), x \in [0, l]$ - berilgan uzluksiz funksiyalar bo'lib, $\alpha_i, \beta_i, i = 1, 2, y_0, y_1$ ushbu $\alpha^2_i + \beta^2_i > 0, i = 1, 2$ shartni qanoatlantiruvchi berilgan haqiqiy sonlar. Bundan tashqari $y(0) = y(+0), y'(0) = y'(+0), y(l) = y(l-0), y'(l) = y'(l-0)$.

Agar $y_0 = y_1 = 0$, ya'ni $l_1(y) = 0, l_2(y) = 0$ bo'lsa, unga bir jinsli chegaraviy shartlar deyiladi. (2.15.1), (2.15.2) ko'rinishdagi masalaga bir jinsli bo'lmagan chegaraviy masala deb ataladi. Ushbu

$$y'' + p_1(x)y' + p_2(x)y = 0, \quad x \in [0, l] \quad (2.15.3)$$

$$l_1(y) = 0, \quad l_2(y) = 0 \quad (2.15.4)$$

ko'rinishdagi masalaga esa bir jinsli chegaraviy masala deyiladi.

2.15.1-teorema. Aytaylik, $y_1(x)$ va $y_2(x)$ funksiyalar (2.15.3) tenglamaning ixtiyoriy chiziqli bog'lanmagan yechimlari bo'lsin. U holda (2.15.3), (2.15.4) bir jinsli chegaraviy masala faqat nol yechimga ega bo'lishi uchun

$$\Delta = \begin{vmatrix} l_1(y_1) & l_1(y_2) \\ l_2(y_1) & l_2(y_2) \end{vmatrix} \neq 0$$

shartning bajarilishi zarur va yetarli.

Isbot. Ma'lumki, (2.15.3) tenglamaning umumiy yechimi

$$y(x) = c_1 y_1(x) + c_2 y_2(x) \quad (2.15.3)$$

ko'rinishga ega. Bunda c_1, c_2 - ixtiyoriy o'zgarmas sonlar. (2.15.5) va (2.15.4) tengliklardan foydalanib quyidagi

$$\begin{cases} c_1 l_1(y_1) + c_2 l_1(y_2) = 0 \\ c_1 l_2(y_1) + c_2 l_2(y_2) = 0 \end{cases} \quad (2.15.6)$$

tenglamalar sistemasini hosil qilamiz. Bu sistema faqat $c_1 = c_2 = 0$ nol yechimga ega bo'lishi uchun $\Delta \neq 0$ bo'lishi zarur va yetarli.

2.15.1-natija. (2.15.3), (2.15.4) bir jinsli chegaraviy masala cheksiz ko'p nolmas yechimga ega bo'lishi uchun $\Delta = 0$ shartning bajarilishi zarur va yetarli.

2.15.1-misol. Ushbu

$$y'' + 2y' + 5y = 0$$

$$l_1(y) = y(0) = 0, \quad l_2(y) = y\left(\frac{\pi}{2}\right) = 0$$

chegaraviy masalani qaraylik. Ko'rinib turibdiki, quyidagi

$$y_1(x) = e^{-x} \cos 2x, \quad y_2(x) = e^{-x} \sin 2x$$

funksiyalar berilgan $y'' + 2y' + 5y = 0$ tenglamaning chiziqli bog'lanmagan yechimlaridan iborat. Chegaraviy shartlardan foydalanib,

$$\Delta = \begin{vmatrix} 1 & 0 \\ -e^{-\frac{\pi}{2}} & 0 \end{vmatrix} = 0$$

ekanligini aniqlaymiz. Bu esa berilgan chegaraviy masala cheksiz ko'p $y(x) = c_2 e^{-x} \sin 2x$ ko'rinishdagi yechimga ega bo'lishini ko'rsatadi. Bunda c_2 - ixtiyoriy o'zgarmas son.

Endi bir jinslimas (2.15.1), (2.15.2) chegaraviy masalaning yechimga egaligi haqidagi masalani o'rganamiz.

Faraz qilaylik, $y_1(x)$ va $y_2(x)$ funksiyalar (2.15.3) bir jinsli tenglamaning chiziqli bog'lanmagan yechimlari bo'lib, $z(x)$ funksiya esa (2.15.1) bir jinslimas tenglamaning biror xususiy yechimi bo'lsin. U holda (2.15.1) differensial tenglamaning umumiy yechimi

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + z(x) \quad (2.15.7)$$

ko‘rinishda bo‘lishi ma’lum. Bunda c_1, c_2 - ixtiyoriy o‘zgarmas sonlar. Bu (2.15.7) ko‘rinishdagi yechimni (2.15.2) bir jinslimas chegaraviy shartlarga qo‘yib, quyidagi

$$\begin{cases} c_1 l_1(y_1) + c_2 l_1(y_2) + l_1(z) = y_0, \\ c_1 l_2(y_1) + c_2 l_2(y_2) + l_2(z) = y_1 \end{cases}$$

tenglamalar sistemasini hosil qilamiz. Bu sistemani ushbu

$$\begin{cases} c_1 l_1(y_1) + c_2 l_1(y_2) = y_0 - l_1(z), \\ c_1 l_2(y_1) + c_2 l_2(y_2) = y_1 - l_2(z) \end{cases} \quad (2.15.8)$$

ko‘rinishda yozib olamiz va quyidagi belgilashlarni kiritamiz:

$$\Delta = \begin{vmatrix} l_1(y_1) & l_1(y_2) \\ l_2(y_1) & l_2(y_2) \end{vmatrix}, \quad A = \begin{vmatrix} l_1(y_1) & l_1(y_2) \\ l_2(y_1) & l_2(y_2) \end{vmatrix},$$

$$B = \begin{vmatrix} l_1(y_1) & l_1(y_2) & y_0 - l_1(z) \\ l_2(y_1) & l_2(y_2) & y_1 - l_2(z) \end{vmatrix}$$

(2.15.8) algebraik tenglamalar sistemasining yechimga egaligi haqidagi Kroneker – Kapelli teoremasidan quyidagi tasdiqning o‘rinli ekanligi kelib chiqadi.

2.15.2-teorema. 1) Agar $\Delta = \det A \neq 0$ bo‘lsa, u holda (2.15.1), (2.15.2) chegaraviy masalaning $[0, l]$ oraliqda aniqlangan yechimi mavjud va yagona bo‘ladi.

2) Agar $\Delta = \det A = 0$ bo‘lib, $\text{rang } A \neq \text{rang } B$ bo‘lsa, u holda (2.15.1), (2.15.2) chegaraviy masala yechimga ega bo‘lmaydi.

3) Agar $\Delta = \det A = 0$ bo‘lib, $\text{rang } A = \text{rang } B$ bo‘lsa, u holda (2.15.1), (2.15.2) chegaraviy masalaning yechimi mavjud, lekin yagona bo‘lmaydi.

2.15.2-natija. (2.15.1), (2.15.2) bir jinslimas chegaraviy masala yagona yechimga ega bo‘lishi uchun (2.15.3), (2.15.4) bir jinsli chegaraviy masala faqat nol yechimga ega bo‘lishi zarur va yetarli.

16-§. Grin funksiyasi

Ushbu

$$y'' + p_1(x)y' + p_2(x)y = f(x), \quad x \in [0, l] \quad (2.16.1)$$

differensial tenglamaga qo‘yilgan

$$\begin{cases} l_1(y) \equiv \alpha_1 y(0) + \beta_1 y'(0) = 0 \\ l_2(y) \equiv \alpha_2 y(l) + \beta_2 y'(l) = 0 \end{cases} \quad (2.16.2)$$

chegaraviy masalani qaraylik. Bunda $p_1(x), p_2(x), f(x), x \in [0, l]$ -berilgan uzluksiz funksiyalar bo'lib, $\alpha_i, \beta_i, i = 1, 2$ ushbu $\alpha_i^2 + \beta_i^2 > 0, i = 1, 2$ shartni qanoatlantiruvchi berilgan haqiqiy sonlar.

Bu paragrafda ushbu

$$y'' + p_1(x)y' + p_2(x)y = 0, \quad (2.16.3)$$

$$l_1(y) = 0, \quad l_2(y) = 0 \quad (2.16.4)$$

bir jinsli chegaraviy masala faqat nol $y(x) \equiv 0$ yechimga ega bo'lgan holda, (2.16.1)+(2.16.2) chegaraviy masala yagona yechimga ega bo'lib, uni Grin funksiyasi yordamida topish mumkinligini ko'rsatamiz.

2.16.1-Ta'rif. (2.16.3)+(2.16.4) chegaraviy masalaning Grin funksiyasi deb, shunday $G(x, t)$ funksiyaga aytiladiki, u

$$P = \{(x, t) \in R^2 : 0 \leq x \leq l, 0 \leq t \leq l\}$$

yopiq sohada aniqlangan bo'lib, quyidagi shartlarni qanoatlantiradi:

1) $G(x, t)$ funksiya P sohada uzluksiz;

2) $t \in [0, l]$ parametrning har bir qiymatida $G(x, t)$ funksiya x o'zgaruvchiga nisbatan $[0, t)$ va $(t, l]$ oraliqlarda (2.16.3) bir jinsli tenglamani qanoatlantiradi;

3) $G(x, t)$ funksiya x o'zgaruvchi bo'yicha birinchi tartibli hosilasining t nuqtadagi sakrashi 1 ga teng:

$$G'_x(x, t)|_{x=t+0} - G'_x(x, t)|_{x=t-0} = 1;$$

4) $G(x, t)$ funksiya x o'zgaruvchi bo'yicha (2.16.2) chegaraviy shartlarni qanoatlantiradi.

2.16.1-teorema. Agar (2.16.3), (2.16.4) bir jinsli chegaraviy masala faqat nol yechimga ega bo'lsa, u holda uning Grin funksiyasi mavjud va yagona bo'lib, u quyidagi

$$G(x, t) = -\frac{1}{W} \begin{cases} \psi(t)\varphi(x), & 0 \leq x \leq t, \\ \varphi(t)\psi(x), & t \leq x \leq l \end{cases}$$

formula yordamida beriladi. Bu yerda $\varphi(x)$ va $\psi(x)$ funksiyalar (2.16.3) tenglamaning mos ravishda ushbu

$$\begin{aligned} \varphi(0) &= \beta_1, \quad \varphi'(0) = -\alpha_1; \\ \psi(l) &= \beta_2, \quad \psi'(l) = -\alpha_2 \end{aligned} \quad (2.16.5)$$

boshlang'ich shartlarni qanoatlantiruvchi yechimlari bo'lib, W esa bu yechimlardan tuzilgan

$$W = W\{\varphi(x), \psi(x)\}$$

Vronskiy determinantiga teng.

Isbot. Avvalo (2.16.3) tenglamaning (2.16.5) boshlang'ich shartlarni qanoatlantiruvchi $\varphi(x)$ va $\psi(x)$ yechimlarini chiziqli bog'lanmaganligini

ko'rsatamiz. Aniqlanishiga ko'ra $\varphi(x)$ (2.16.2) chegaraviy shartlarning birinchisini, $\psi(x)$ esa ikkinchisini qanoatlantiradi. Agarda $\varphi(x)$ va $\psi(x)$ yechimlar chiziqli bog'liq bo'lsa, u holda $\varphi(x) = c\psi(x)$ bo'lib, $\psi(x)$ yechim ($\psi(x) \neq 0$, chunki, $\alpha_2^2 + \beta_2^2 > 0$) ikkala chegaraviy shartni ham qanoatlantiradi. Bu esa teoremaning shartiga zid. Shuning uchun $\varphi(x)$ va $\psi(x)$ funksiyalar (2.16.3) tenglamaning fundamental yechimlari sistemasini tashkil qiladi. Bunga ko'ra (2.16.3) tenglamaning ixtiyoriy yechimi

$$y = c_1\varphi(x) + c_2\psi(x)$$

ko'rinishda yoziladi. Shuning uchun Grin funksiyasini ushbu

$$G(x,t) = \begin{cases} A(t)\varphi(x) + B(t)\psi(x), & x \leq t, \\ C(t)\varphi(x) + D(t)\psi(x), & x \geq t \end{cases} \quad (2.16.6)$$

ko'rinishda izlaymiz. Grin funksiyasi ta'rifining birinchi shartiga ko'ra u $x = t$ nuqtada uzluksiz bo'ladi. Demak quyidagi

$$A(t)\varphi(t) + B(t)\psi(t) - C(t)\varphi(t) - D(t)\psi(t) = 0 \quad (2.16.7)$$

tenglik bajariladi. Uchinchi shartga ko'ra

$$A(t)\varphi'(t) + B(t)\psi'(t) - C(t)\varphi'(t) - D(t)\psi'(t) = 1 \quad (2.16.8)$$

munosabat o'rinli. To'rtinchi shartlardan foydalanib

$$[A(t)\varphi(0) + B(t)\psi(0)]\alpha_1 + [A(t)\varphi'(0) + B(t)\psi'(0)]\beta_1 = 0, \quad (2.16.9)$$

$$[C(t)\varphi(l) + D(t)\psi(l)]\alpha_2 + [C(t)\varphi'(l) + D(t)\psi'(l)]\beta_2 = 0 \quad (2.16.10)$$

tengliklarni topamiz. Bu tengliklarni quyidagi ko'rinishda yozib olamiz:

$$A(t)[\alpha_1\varphi(0) + \beta_1\varphi'(0)] + B(t)[\alpha_1\psi(0) + \beta_1\psi'(0)] = 0, \quad (2.16.11)$$

$$C(t)[\alpha_2\varphi(l) + \beta_2\varphi'(l)] + D(t)[\alpha_2\psi(l) + \beta_2\psi'(l)] = 0. \quad (2.16.12)$$

Agar ushbu

$$W = \begin{vmatrix} \varphi(x) & \psi(x) \\ \varphi'(x) & \psi'(x) \end{vmatrix}_{x=0} = \varphi(0)\psi'(0) - \psi(0)\varphi'(0) = \beta_1\psi'(0) + \alpha_1\psi(0),$$

$$W = \begin{vmatrix} \varphi(x) & \psi(x) \\ \varphi'(x) & \psi'(x) \end{vmatrix}_{x=l} = \varphi(l)\psi'(l) - \psi(l)\varphi'(l) = \beta_2\psi'(l) + \alpha_2\psi(l)$$

formulalarni va boshlang'ich shartlarni hisobga olsak, (2.16.11) va (2.16.12) tengliklar quyidagi ko'rinishni oladi:

$$\begin{aligned} B(t) \cdot W &= 0, \\ C(t) \cdot W &= 0. \end{aligned} \quad (2.16.13)$$

Quyidagi ikki holni ko'rib chiqamiz:

1) $W \neq 0$ bo'lsin, u holda (2.16.13) tenglamalardan

$$B(t) = 0, \quad C(t) = 0 \quad (2.16.14)$$

kelib chiqadi. Bularni (2.16.7) va (2.16.8) tengliklarga qo'ysak, ushbu

$$\begin{cases} A\varphi(t) - D\psi(t) = 0, \\ A\varphi'(t) - D\psi'(t) = 1. \end{cases} \quad (2.16.15)$$

tenglamalar sistemasini hosil qilamiz. Bu sistemani Kramer qoidasi yordamida yechamiz:

$$\begin{aligned} \Delta &= \begin{vmatrix} \varphi(t) & -\psi(t) \\ \varphi'(t) & -\psi'(t) \end{vmatrix} = -\varphi(t)\psi'(t) + \varphi'(t)\psi(t) = -W, \\ \Delta_1 &= \begin{vmatrix} 0 & -\psi(t) \\ 1 & -\psi'(t) \end{vmatrix} = \psi(t), & \Delta_2 &= \begin{vmatrix} \varphi(t) & 0 \\ \varphi'(t) & 1 \end{vmatrix} = \varphi(t), \\ A(t) &= -\frac{\psi(t)}{W}, & D(t) &= -\frac{\varphi(t)}{W}. \end{aligned} \quad (2.16.16)$$

Topilgan (2.16.14) va (2.16.16) ifodalarni (2.16.6) tenglikka qo'yib,

$$G(x,t) = \begin{cases} -\frac{\varphi(x)\psi(t)}{W}, & x \leq t \\ -\frac{\psi(x)\varphi(t)}{W}, & x \geq t \end{cases} \quad (2.16.17)$$

formulani topamiz. Demak, bu holda Grin funksiyasi mavjud va u (2.16.17) formula bilan beriladi.

2) Faraz qilaylik $W = 0$ bo'lsin, u holda $\psi(t) = \gamma\varphi(t)$ bo'ladi. Buni (2.16.7) va (2.16.8) tengliklarga qo'yib,

$$\begin{aligned} (A + B\gamma - C - D\gamma)\varphi(t) &= 0, \\ (A + B\gamma - C - D\gamma)\varphi'(t) &= 1 \end{aligned}$$

bo'lishini topamiz. $\varphi(t) \neq 0$ bo'lgani uchun

$$A + B\gamma - C - D\gamma = 0$$

bo'ladi. Buni yuqoridagi tengliklarning ikkinchisiga qo'ysak,

$$0 \cdot \varphi'(t) = 1$$

ziddiyat kelib chiqadi. Demak, bu holda Grin funksiyasi mavjud emas ekan.

2.16.1-natija. Grin funksiyasi uchun yozilgan (2.16.17) formuladan uning x va t o'zgaruvchilarga nisbatan simmetrikligi, ya'ni

$$G(x,t) = G(t,x) \quad (2.16.18)$$

kelib chiqadi.

2.16.2-teorema. (D. Gilbert). Agar (2.16.3)+(2.16.4) bir jinsli chegaraviy masala faqat nol yechimga ega bo'lsa, u holda ixtiyoriy $f(x) \in C[0,l]$ funksiya uchun (2.16.1)+(2.16.2) bir jinsli bo'lmagan chegaraviy masalaning yechimi mavjud va yagona bo'ladi, hamda u ushbu

$$y(x) = \int_0^l G(x,t)f(t)dt \quad (2.16.19)$$

formula orqali topiladi. Bu yerda $G(x,t)$ (2.16.3)+(2.16.4) chegaraviy masalaning Grin funksiyasi.

Isbot. Avvalo (2.16.19) formula bilan aniqlangan $y(x)$ funksiya (2.16.1)+(2.16.2) chegaraviy masalaning yechimi ekanligini tekshirib ko'ramiz. Buning uchun uni ushbu

$$\begin{aligned} y(x) &= \int_0^x G(x,t) f(t) dt + \int_x^l G(x,t) f(t) dt = \\ &= -\frac{\psi(x)}{W} \int_0^x \varphi(t) f(t) dt - \frac{\varphi(x)}{W} \int_x^l \psi(t) f(t) dt \end{aligned} \quad (2.16.20)$$

ko'rinishda yozib olamiz va uning hosilalarini hisoblaymiz:

$$\begin{aligned} y'(x) &= -\frac{\psi'(x)}{W} \int_0^x \varphi(t) f(t) dt - \frac{\psi(x)\varphi(x)}{W} f(x) - \\ &\quad - \frac{\varphi'(x)}{W} \int_x^l \psi(t) f(t) dt + \frac{\psi(x)\varphi(x)}{W} f(x) = \\ &= -\frac{\psi'(x)}{W} \int_0^x \varphi(t) f(t) dt - \frac{\varphi'(x)}{W} \int_x^l \psi(t) f(t) dt; \end{aligned} \quad (2.16.21)$$

$$\begin{aligned} y''(x) &= -\frac{\psi''(x)}{W} \int_0^x \varphi(t) f(t) dt - \frac{\psi'(x)\varphi(x)}{W} f(x) - \\ &\quad - \frac{\varphi''(x)}{W} \int_x^l \psi(t) f(t) dt + \frac{\psi(x)\varphi'(x)}{W} f(x) \end{aligned} \quad (2.16.22)$$

Ushbu

$$\begin{aligned} \varphi''(x) &= -p_1(x)\varphi'(x) - p_2(x)\varphi(x), \\ \psi''(x) &= -p_1(x)\psi'(x) - p_2(x)\psi(x), \\ W &= \varphi(x)\psi'(x) - \psi(x)\varphi'(x) \end{aligned}$$

ayniyatlardan va (2.16.20), (2.16.21) tengliklardan foydalanib, (2.16.22) formuladan quyidagi tenglamani keltirib chiqaramiz:

$$\begin{aligned} y''(x) &= -\frac{p_1(x)\psi'(x) + p_2(x)\psi(x)}{W} \int_0^x \varphi(t) f(t) dt - \\ &\quad - \frac{p_1(x)\varphi'(x) + p_2(x)\varphi(x)}{W} \int_x^l \psi(t) f(t) dt + \frac{\varphi'(x)\psi(x) - \psi'(x)\varphi(x)}{W} f(x) = \end{aligned}$$

$$\begin{aligned}
&= -p_1(x) \frac{\psi'(x)}{W} \int_0^x \varphi(t) f(t) dt - p_2(x) \frac{\psi(x)}{W} \int_0^x \varphi(t) f(t) dt - \\
&- p_1(x) \frac{\varphi'(x)}{W} \int_x^l \psi(t) f(t) dt - p_2(x) \frac{\varphi(x)}{W} \int_x^l \psi(t) f(t) dt + f(x) = \\
&= -p_1(x) \left[\frac{\psi'(x)}{W} \int_0^x \varphi(t) f(t) dt + \frac{\varphi'(x)}{W} \int_x^l \psi(t) f(t) dt \right] - \\
&- p_2(x) \left[\frac{\psi(x)}{W} \int_0^x \varphi(t) f(t) dt + \frac{\varphi(x)}{W} \int_x^l \psi(t) f(t) dt \right] + f(x) = \\
&= -p_1(x) y'(x) - p_2(x) y(x) + f(x).
\end{aligned}$$

Endi (2.16.2) chegaraviy shartlar bajarilishini ko'rsatamiz. Buning uchun (2.16.20) va (2.16.21) tengliklardan foydalanib, avvalo $l_1(y) := \alpha_1 y(0) + \beta_1 y'(0) = 0$ ekanligini ko'rsatamiz:

$$\begin{aligned}
y(0) &= -\frac{\varphi(0)}{W} \int_0^l \psi(t) f(t) dt, \\
y'(0) &= -\frac{\varphi'(0)}{W} \int_0^l \psi(t) f(t) dt, \\
l_1(y) &:= \alpha_1 y(0) + \beta_1 y'(0) = -\frac{\alpha_1 \varphi(0) + \beta_1 \varphi'(0)}{W} \int_0^l \psi(t) f(t) dt = \\
&= -\frac{\alpha_1 \beta_1 - \alpha_1 \beta_1}{W} \int_0^l \psi(t) f(t) dt = 0.
\end{aligned}$$

Xuddi shuningdek $l_2(y) := \alpha_2 y(l) + \beta_2 y'(l) = 0$ bo'lishini ham ko'rsatish mumkin.

Yechimning yagonaligini ko'rsatish uchun (2.16.1)+(2.16.2) chegaraviy masalaning ikkita $y_1(x) \neq y_2(x)$ yechimi mavjud deb faraz qilamiz. $y_1(x)$ va $y_2(x)$ yechimlarni (2.16.1)+(2.16.2) chegaraviy masalaga qo'yib, hosil bo'lgan ayniyatlarni mos ravishda bir – biridan ayirsak, hamda $y(x) = y_1(x) - y_2(x)$ belgilash kiritsak, u holda $y(x)$ funksiya (2.16.3) + (2.16.4) bir jinsli chegaraviy masalani qanoatlantirishini ko'ramiz. Teorema shartiga ko'ra u faqat $y(x) \equiv 0$ nol yechimga ega. Bundan $y_1(x) \equiv y_2(x)$ kelib chiqadi.

2.16.1-misol. Quyidagi

$$y'' + y = f(x), \quad y(0) = 0, \quad y'(\pi) = 0$$

chegaraviy masalaning yechimini Grin funksiyasidan foydalanib toping.

Yechish. Avvalo ushbu

$$y'' + y = 0, \quad y(0) = 0, \quad y'(\pi) = 0$$

bir jinsli chegaraviy masalani qaraylik. Berilgan $y'' + y = 0$ tenglamaning umumiy yechimi

$$y(x) = c_1 \cos x + c_2 \sin x$$

ko‘rinishda bo‘ladi. Chegaraviy shartlardan foydalanib c_1 va c_2 o‘zgarmlarining qiymatlarini aniqlaymiz:

$$y(0) = 0 \Rightarrow c_1 = 0; \quad y'(\pi) = 0 \Rightarrow c_2 = 0.$$

Demak, bir jinsli chegaraviy masala faqat $y(x) \equiv 0$ yechimga ega ekan. Bundan ko‘rinadiki bir jinsli chegaraviy masalaning Grin funksiyasi mavjud va yagona bo‘ladi.

Endi Grin funksiyasining ta‘rifidan foydalanib, uni

$$G(x, t) = \begin{cases} a(t) \sin x, & x \leq t, \\ b(t) \cos x, & x \geq t \end{cases}$$

ko‘rinishda izlaymiz. Grin funksiyasi $x = t$ nuqtada uzluksiz bo‘lgani uchun

$$a(t) \sin t - b(t) \cos t = 0$$

munosabatga, $G'(x, t)$ hosilaning $x = t$ nuqtada birinchi tur uzulishga ega bo‘lib, sakrash uzunligi 1 ga teng bo‘lganligi uchun

$$-b(t) \sin t - a(t) \cos t = 1$$

tenglikka ega bo‘lamiz. Bu ikki algebraik tenglamalar sistemasini yechib

$$a(t) = -\cos t, \quad b(t) = -\sin t$$

funksiyalarni topib olamiz. Shunday qilib, bir jinsli chegaraviy masalaning Grin funksiyasi ushbu

$$G(x, t) = \begin{cases} -\cos t \sin x, & 0 \leq x \leq t \\ -\sin t \cos x, & t \leq x \leq \pi \end{cases}$$

formula orqali aniqlanar ekan. Bundan foydalanib berilgan bir jinsli bo‘lmagan chegaraviy masalaning yechimini ham topishimiz mumkin:

$$y(x) = \int_0^{\pi} G(x, t) f(t) dt.$$

2.16.2-Misol. Ushbu

$$y'' = f(x), \quad y(0) = 0, \quad y(\pi) = 0$$

chegaraviy masalaning yechimini Grin funksiyasi yordamida toping.

Yechish. Avvalo quyidagi

$$y'' = 0, \quad y(0) = 0, \quad y(\pi) = 0$$

bir jinsli chegaraviy masalani qaraymiz. Ushbu $y_1(x) = 1$, $y_2(x) = x$ funksiyalar $y'' = 0$ bir jinsli differensial tenglamaning chiziqli erkli yechimlari bo‘lgani uchun, uning umumiy yechimi $y(x) = c_1 + c_2 x$

ko‘rinishda bo‘ladi. Bu yerda c_1, c_2 - ixtiyoriy sonlar. Endi berilgan $y(0)=0, y(\pi)=0$ chegaraviy shartlardan foydalansak $c_1=0, c_2=0$ ekanligini topamiz. Bundan ko‘rinadiki qaralayotgan bir jinsli chegaraviy masala faqat $y(x)=0$ yechimga ega bo‘ladi. Shuning uchun uning Grin funksiyasi mavjud va yagonadir.

Izlanayotgan Grin funksiyasini

$$G(x,t) = \begin{cases} c_1(t) \cdot 1 + c_2(t) \cdot x, & x \leq t \\ a_1(t) \cdot 1 + a_2(t) \cdot t, & x > t. \end{cases}$$

ko‘rinishda izlaymiz. Ta‘rifga ko‘ra quyidagi

$$\begin{cases} c_1(t) + c_2(t) \cdot t = a_1(t) + a_2(t) \cdot t, \\ a_2(t) - c_2(t) = 1, \\ c_1(t) = 0, \\ a_1(t) + a_2(t) \cdot \pi = 0 \end{cases}$$

algebraik tenglamalar sistemasiga ega bo‘lamiz. Bu sistemani yechib

$$c_1(t) = 0, c_2(t) = \frac{t-\pi}{\pi}, a_1(t) = -t, a_2(t) = \frac{t}{\pi}$$

noma‘lumlarni aniqlaymiz. Natijada, ushbu

$$G(x,t) = -\frac{1}{\pi} \begin{cases} (\pi-t) \cdot x, & x \leq t \\ (\pi-t) \cdot t, & x \geq t \end{cases}$$

ko‘rinishdagi Grin funksiyasiga ega bo‘lamiz. Berilgan bir jinsli bo‘lmagan chegaraviy masalaning yechimini quyidagi

$$y(x) = \int_0^{\pi} G(x,t) f(t) dt$$

formuladan foydalanib topish mumkin.

2.16.3-misol. Quyidagi

$$y'' = f(x), \alpha_1 y(0) + \beta_1 y'(0) = 0; \alpha_2 y(l) + \beta_2 y'(l) = 0$$

chegaraviy masalaning yechimini Grin funksiyasidan foydalanib toping.

Bunda $\alpha_i, \beta_i, i=1,2$ sonlar $\alpha_i^2 + \beta_i^2 > 0, i=1,2$ shartni qanoatlantiradi.

Mustaqil yechish uchun mashqlar [8], §17, №711-723; [13], II-Bob, §10, №12-34, №40-50.

III BOB. DIFFERENSIAL TENGLAMANI INTEGRALLASHDA QATORLARDAN FOYDALANISH.

1-§. Birinchi tartibli hosilaga nisbatan yechilgan differensial tenglamaning golomorf yechimi.

1. 3.1.1-ta'rif. Agar $f(x)$ funksiyani $x_0 \in R$ nuqtaning biror atrofida

$$f(x) = C_0 + \sum_{k=1}^{\infty} C_k (x - x_0)^k, |x - x_0| < \rho$$

darajali qatorga yoyish mumkin bo'lsa, $f(x)$ ga $x = x_0$ nuqtada golomorf deyiladi.

Ushbu

$$y' = f(x, y) \quad (3.1.1)$$

$$y(0) = 0 \quad (3.1.2)$$

Koshi masalasini qaraylik. Chunki umumiy ko'rinishdagi

$$y' = f(x, y), y(x_0) = y_0$$

Koshi masalasini

$$x - x_0 = \xi, y - y_0 = \eta$$

almashtirish yordamida (3.1.1), (3.1.2) ko'rinishga keltirish mumkin.

3.1.1-teorema (Koshi). Agar $(0,0)$ nuqtada $f(x, y)$ funksiya golomorf, ya'ni

$$f(x, y) = \sum_{m,n=0}^{\infty} a_{m,n} x^m y^n, |x| < \rho, |y| < r \quad (3.1.3)$$

bo'lsa, u holda (3.1.1), (3.1.2) Koshi masalasining $x=0$ nuqtada golomorf bo'lgan

$$y(x) = \sum_{k=1}^{\infty} C_k x^k, |x| < \rho_1, |\rho_1| < \rho \quad (3.1.4)$$

yagona yechimi mavjud.

Isbot. 1) Formal yechimni qurish algoritmi. Berilgan (3.1.1), (3.1.2) Koshi masalasining yechimini, ushbu

$$y(x) = \sum_{k=1}^{\infty} C_k x^k, C_0 = y(0) = 0 \quad (3.1.5)$$

ko'rinishda izlaymiz. Bu yerda C_k -hozircha noma'lum sonlar. (3.1.1) differensial tenglamani (3.1.3) tasvirdan foydalanib quyidagi ko'rinishda yozamiz:

$$y'(x) = \sum_{m,n=0}^{\infty} a_{mn} x^m y^n. \quad (3.1.6)$$

Endi (3.1.5) qatorni (3.1.6) tenglikka qo'yib,

$$\sum_{k=1}^{\infty} kc_k x^{k-1} = \sum_{m,n=0}^{\infty} a_{mn} x^m \left(\sum_{k=1}^{\infty} c_k x^k \right)^n = \sum_{k=0}^{\infty} d_k x^k \quad (3.1.7)$$

munosabatni hosil qilamiz. Bu tenglikda darajali qatorlarning yoki golomorf funksiyalarning yagonaligi haqidagi teorema asoslanib, x o'zgaruvchining mos darajalari oldidagi koeffitsiyentlarni tenglashtirish natijasida quyidagi rekkurent tengliklarni hosil qilamiz:

$$1 \cdot C_1 = a_{00} \Rightarrow C_1 = a_{00},$$

$$2 \cdot C_2 = a_{10} + a_{01}C_1 \Rightarrow C_2 = \frac{1}{2}(a_{10} + a_{01}a_{00}) \equiv P_2(a_{\lambda\mu})$$

.....

Xuddi shuningdek x^{k-1} oldidagi koeffitsiyentlarni tenglashtirib

$$C_k = P_k(a_{\lambda\mu}) \quad (3.1.8)$$

tasvirni topamiz. Bu yerda P_k -musbat koeffitsiyentli ko'phad (P_2 -da bu koeffitsiyent $\frac{1}{2}$ ga teng).

Shunday qilib, agar golomorf yechim mavjud bo'lsa, u yagona bo'lar ekan. Chunki (3.1.5) formal yechimning C_k koeffitsiyentlari (3.1.8) formula orqali yagona aniqlanadi.

2) Teorema isbotining asosiy qismi, yechimni ifodalovchi (3.1.5) darajali qatorni $x=0$ nuqtaning biror atrofida yaqinlashuvchi ekanligini ko'rsatishdan iborat. Buning uchun $x=0$ nuqtaning biror $|x| < \rho_1$ atrofida yaqinlashuvchi majarant

$$\sum_{k=1}^{\infty} \overline{C}_k x^k, \quad |C_k| \leq \overline{C}_k, \quad k \in N \quad (3.1.9)$$

musbat hadli ($\overline{C}_k > 0$) darajali qatorni tuzish yetarli.

Yuqoridagi g'oyani amalga oshirish maqsadida berilgan $f(x, y)$ funksiyaning biror $F(x, y)$ majarantasini quyidagi sxema yordamida tuzamiz.

(3.1.3) tenglikning o'ng tomonidagi qator $P = \{(x, y) \in R^2 : |x| < \rho, |y| < r\}$ to'g'ri to'rtburchakda yaqinlashuvchi bo'lgani uchun, ushbu $0 < \rho' < \rho, 0 < r' < r$ tengsizliklarni qanoatlantiruvchi istalgan ρ', r' sonlar uchun

$$|a_{mn}| \leq \frac{M}{(\rho')^m (r')^n} \equiv A_{mn} \quad (3.1.10)$$

Koshi tengsizligi o'rinli. Bu yerda

$$M = \sum_{m,n=0}^{\infty} |a_{mn}| (\rho')^m (r')^n. \quad (3.1.11)$$

(3.1.10) tengsizlikning o'ng tomoni orqali aniqlangan A_{mn} -sonlardan foydalanib, quyidagi darajali qatorni tuzamiz:

$$\begin{aligned} \sum_{m,n=0}^{\infty} A_{m,n} x^m y^n &= \sum_{m,n=0}^{\infty} \frac{M}{(\rho')^m (r')^n} x^m y^n = \sum_{m,n=0}^{\infty} M \left(\frac{x}{\rho'}\right)^m \left(\frac{y}{r'}\right)^n = \\ &= \frac{M}{\left(1 - \frac{x}{\rho'}\right)\left(1 - \frac{y}{r'}\right)} \equiv F(x, y), \quad |x| < \rho', \quad |y| < r'. \end{aligned}$$

Shunday qilib, $f(x, y)$ funksiyaning majarantasi sifatida ushbu

$$F(x, y) = \frac{M}{\left(1 - \frac{x}{\rho'}\right)\left(1 - \frac{y}{r'}\right)}, \quad |x| < \rho', \quad |y| < r' \quad (3.1.12)$$

funksiyani olish mumkin.

Endi berilgan (3.1.1), (3.1.2) Koshi masalasi o'rniga yordamchi

$$\frac{d\bar{y}}{dx} = F(x, \bar{y}), \quad \bar{y}(0) = 0,$$

ya'ni

$$\frac{d\bar{y}}{dx} = \frac{M}{\left(1 - \frac{x}{\rho'}\right)\left(1 - \frac{\bar{y}}{r'}\right)}, \quad \bar{y}(0) = 0 \quad (3.1.13)$$

majarant Koshi masalasini qaraymiz. Bu masala yagona aniq yechimga ega. Chunki (3.1.13) o'zgaruvchilarga ajraladigan differensial tenglamadir. Avvalo bu differensial tenglamani integrallab, uning umumiy yechimini topamiz:

$$\begin{aligned} \left(1 - \frac{\bar{y}}{r'}\right) d\bar{y} &= \frac{M}{1 - \frac{x}{\rho'}} dx, \\ -\frac{r'}{2} \left(1 - \frac{\bar{y}}{r'}\right)^2 &= -M \rho' \ln \left(1 - \frac{x}{\rho'}\right) + C, \quad C = const. \end{aligned}$$

Ushbu $\bar{y}(0) = 0$ boshlang'ich shartdan foydalanib, C -o'zgaruvchining qiymatini topamiz:

$$C = -\frac{r'}{2}.$$

Natijada (3.1.13) Koshi masalasining yechimini hosil qilamiz:

$$-\frac{r'}{2} \left(1 - \frac{\bar{y}}{r'}\right)^2 = -M \rho' \ln \left(1 - \frac{x}{\rho'}\right) - \frac{r'}{2}.$$

Bu tenglikning ikki tomonini $\left(-\frac{2}{r'}\right)$ ga ko'patirib, quyidagi

$$\left(1 - \frac{\bar{y}}{r'}\right)^2 = \frac{2M\rho'}{r'} \ln\left(1 - \frac{x}{\rho'}\right) + 1,$$

ya'ni

$$\bar{y}(x) = r' \left(1 - \sqrt{1 + \frac{2M\rho'}{r'} \ln\left(1 - \frac{x}{\rho'}\right)} \right) \quad (3.1.14)$$

yechimni topamiz. Bu yechim $x=0$ nuqtada golomorf. Chunki golomorf funksiyalarning superpazitsiyasi (murakkab funksiyasi)

$$\sqrt{1+z}, \quad z = \frac{2M\rho'}{r'} \ln\left(1 - \frac{x'}{\rho'}\right)$$

yana golomorf. Shunday qilib, (3.1.14) yechimni ushbu

$$\bar{y}(x) = \sum_{k=1}^{\infty} \bar{C}_k x^k \quad (3.1.15)$$

ko'rinishda tasvirlash mumkin. Endi (3.1.15) darajali qatorning yaqinlashish sohasini aniqlaymiz. Buning uchun, uning yaqinlashish radiusini baholaymiz. Binomial va logarifmik qatorlarning yaqinlashish radiusi 1 ga teng bo'lgani uchun Abel teoremasiga asosan x ning musbat qiymatlari bilan cheklanish yetarli. Shunday qilib x ning qabul qiladigan qiymatlari

$$\begin{cases} 0 < x < \rho', \\ -\frac{2M\rho'}{r'} \ln\left(1 - \frac{x}{\rho'}\right) < 1 \end{cases} \quad (3.1.16)$$

tengsizliklar sistemasini qanoatlantiradi. Bu sistemaning ikkinchi tengsizligini yechib quyidagi

$$\ln\left(1 - \frac{x}{\rho'}\right) > -\frac{r'}{2M\rho'},$$

ya'ni

$$x < \rho' \left(1 - e^{-\frac{r'}{2M\rho'}} \right)$$

bahoni olamiz. Shunday qilib (3.1.15) darajali qator ushbu

$$|x| < \rho_1, \quad \rho_1 = \rho' \left(1 - e^{-\frac{r'}{2M\rho'}} \right)$$

sohada yaqinlashadi.

Endi (3.1.15) qatorning barcha \bar{C}_k koeffitsiyentlarining musbatligini va (3.1.9) tengsizlikning bajarilishini ko'rsatish lozim. Aniqmas koeffitsiyentlar usulidan foydalanib \bar{C}_k ni quyidagi

$$\bar{C}_k = P_k(A_{\lambda\mu})$$

formula orqali topish mumkin. P_k ko'phadning argumentlari majarant $F(x,y)$ funksiyaning yoyilmasining koeffitsiyentlaridan iborat. Bu P_k ko'phad koeffitsiyentlarining musbatligidan hamda (3.1.10) Koshi tengsizligidan \overline{C}_k larning musbatligi va (3.1.9) tengsizlikning o'rinli bo'lishi kelib chiqadi.

3.1.1-misol. Ushbu

$$y' = y^2, y(0) = 1$$

Koshi masalasining golomorf yechimini toping.

Yechish: Berilgan differensial tenglamada o'zgaruvchilarni ajratib uning umumiy yechimini topamiz:

$$y(x) = -\frac{1}{x+c}.$$

Ushbu $y(0)=1$ boshlang'ich shartdan $C=-1$ ekanligini aniqlaymiz. So'ngra berilgan Koshi masalasining yechimi

$$y(x) = -\frac{1}{x-1} = \frac{1}{1-x} \quad (-\infty < x < 1)$$

topiladi. Bu yechimni quyidagi

$$y(x) = \sum_{k=1}^{\infty} x^k, |x| < 1$$

ko'rinishda yozish mumkin.

2-§. Birinchi tartibli chiziqli differensial tenglamaning golomorf yechimi

Ushbu

$$\frac{dy}{dx} = p(x)y + q(x), \quad (3.2.1)$$

$$y(0) = 0 \quad (3.2.2)$$

Koshi masalasini qaraylik. Bu yerda $p(x)$ va $q(x)$ funksiyalar $x=0$ nuqtada golomorf:

$$p(x) = \sum_{k=0}^{\infty} p_k x^k \quad q(x) = \sum_{k=0}^{\infty} q_k x^k \quad |x| < \rho. \quad (3.2.3)$$

3.2.1-teorema. (3.2.1) - (3.2.3) Koshi masalasining $x=0$ nuqtada golomorf

$$y(x) = \sum_{k=1}^{\infty} c_k x^k \quad (3.2.4)$$

yechimi mavjud va yagona. Bunda (3.2.4) darajali qator

$$|x| < \rho \quad (3.2.5)$$

sohada yaqinlashadi.

Isbot. Avvalo, aniqmas koeffitsiyentlar usulidan foydalanib (3.2.4) formal yechimni qurib olamiz va c_k koeffitsiyentlarini ushbu

$$c_k = p_k(p_\lambda, p_\mu) \quad (3.2.6)$$

formula orqali yagona aniqlash mumkinligi xuddi oldingi paragrafdagidek amalga oshiriladi. Bundan keyin (3.2.4) qatorning yaqinlashishini ko'rsatamiz. Buning uchun yordamchi majarant Koshi masalasini tuzib olamiz. Shu maqsadda

$$f(x, y) = p(x)y + q(x)$$

funksiyaning $F(x, y)$ majarant funksiyasini tuzish lozim bo'ladi. Avvalo $p(x)$ va $q(x)$ funksiyalarning umumiy majarantasini, ya'ni

$$\Phi(x) \equiv \frac{M}{1 - \frac{x}{\rho'}}, \quad |x| < \rho'$$

olamiz. Bu yerda $M > 0$ son. Haqiqatan ham $0 < \rho' < \rho$ tengsizlikni qanoatlantiruvchi ρ' sonini olib ushbu

$$\sum_{k=0}^{\infty} |p_k| (\rho')^k = M_1, \quad \sum_{k=0}^{\infty} |q_k| (\rho')^k = M_2,$$

belgilashlarni kiritsak, u holda Koshi tengsizligi

$$|p_k| \leq \frac{M_1}{(\rho')^k}, \quad |q_k| \leq \frac{M_2}{(\rho')^k},$$

ko'rinishni oladi.

Quyidagi

$$\sum_{k=0}^{\infty} \frac{M_1}{(\rho')^k} x^k = \frac{M_1}{1 - \frac{x}{\rho'}}, \quad \sum_{k=0}^{\infty} \frac{M_2}{(\rho')^k} x^k = \frac{M_2}{1 - \frac{x}{\rho'}}$$

qatorlar (3.2.3) darajali qatorlar uchun majoranta vazifasini o'taydi. Shuning uchun $M = \max(M_1, M_2)$ deb,

$$F(x, y) = \Phi(x)y + \Phi(x) = \Phi(x)(y + 1),$$

ya'ni

$$F(x, y) = \frac{M}{1 - \frac{x}{\rho'}} (y + 1), \quad |x| < \rho'$$

funksiyani berilgan $f(x, y) = p(x)y + q(x)$ ning majarantasi sifatida olamiz.

Shunday qilib, majarant Koshi masalasi sifatida ushbu

$$\frac{d\bar{y}}{dx} = \frac{M}{1 - \frac{x}{\rho'}} (\bar{y} + 1), \quad \bar{y}(0) = 0 \quad (3.2.8)$$

Koshi masalasini olamiz. Endi bu masalaning yechimini topish bilan shug'ullanamiz:

$$\frac{d\bar{y}}{\bar{y}+1} = \frac{M}{1-\frac{x}{\rho'}},$$

$$\ln(\bar{y}+1) = -M\rho' \ln\left(1-\frac{x}{\rho'}\right) + C, \quad C = \text{const.}$$

Ushbu $\bar{y}(0) = 0$ boshlang'ich shartdan foydalanib C -o'zgarishning qiymatini aniqlaymiz:

$$C = 0.$$

Demak, quyidagi

$$\bar{y}(x) = -1 + \left(1 - \frac{x}{\rho'}\right)^{-M\rho'} \quad (3.2.9)$$

funksiya (3.2.8) majarant Koshi masalasining yechimidan iborat bo'lar ekan. Topilgan (3.2.9) yechimning $x=0$ nuqtada holomorfligi ekanligi ravshan. Shuning uchun

$$\bar{y}(x) = \sum_{k=1}^{\infty} \bar{C}_k x^k, \quad |x| < \rho' \quad (3.2.10)$$

o'rinli. Yuqoridagi (3.2.6) tenglikdan foydalanib,

$$\bar{C}_k > 0, \quad |\bar{C}_k| \leq \bar{C}_k \quad (k=1,2,\dots)$$

ekanligini ham ko'rsatish mumkin. Shuning uchun (3.2.10) qator (3.2.4) uchun majarant qator vazifasini bajaradi. Bundan esa (3.2.4) qatorning $|x| < \rho'$ sohada yaqinlashuvchiligi kelib chiqadi. Ammo ρ' ni ρ ga istalgancha yaqin qilib tanlash mumkinligini hisobga olsak (3.2.4) qatorning $|x| < \rho$ sohada ham yaqinlashuvchiligi kelib chiqadi.

3-§. Ikkinchi tartibli bir jinsli chiziqli differensial tenglamaning holomorfligi yechimi

3.3.1-teorema. Agar ushbu

$$y'' + p(x)y' + q(x)y = 0 \quad (3.3.1)$$

differensial tenglamaga qo'yilgan

$$y(x_0) = y_0, \quad y'(x_0) = y'_0 \quad (3.3.2)$$

Koshi masalasida $p(x)$, $q(x)$ funksiyalar $x = x_0$ nuqtada holomorfligi, ya'ni

$$p(x) = \sum_{k=0}^{\infty} p_k (x-x_0)^k, \quad q(x) = \sum_{k=0}^{\infty} q_k (x-x_0)^k, \quad |x-x_0| < \rho \quad (3.3.3)$$

bo'lsa, u holda (3.3.1), (3.3.2) masalasining

tenglamalar sistemasini hosil qilamiz. Bundan foydalanib, C_k - koeffitsiyentlarni boshlang'ich shartlar va p_k, q_k koeffitsiyentlar yordamida yagona aniqlash mumkin:

$$\begin{aligned} C_0 &= y_0, \\ C_1 &= y_0', \\ C_2 &= -\frac{1}{2}(p_0 y_0' + q_0 y_0), \\ C_3 &= \frac{1}{3}[p_0(p_0 y_0' + q_0 y_0) - y_0'(p_1 + q_0) - y_0 q_1], \end{aligned} \quad (3.3.11)$$

Berilgan (3.3.1) differensial tenglamaning

$$y_1(0) = 1, y_1'(0) = 0 \quad (3.3.12)$$

boshlang'ich shartlarni qanoatlantiruvchi yechimi

$$y_1(x) = 1 + \sum_{k=2}^{\infty} C_k^{(1)} (x - x_0)^k, \quad |x - x_0| < \rho \quad (3.3.13)$$

ko'rinishda bo'ladi. Bundan tashqari (3.3.1) differensial tenglamaning ushbu

$$y_2(0) = 1, y_2'(0) = 0 \quad (3.3.14)$$

boshlang'ich shartlarni qanoatlantiruvchi yechimi

$$y_2(x) = x - x_0 + \sum_{k=2}^{\infty} C_k^{(2)} (x - x_0)^k, \quad |x - x_0| < \rho \quad (3.3.15)$$

ko'rinishda bo'ladi.

(3.3.1) differensial tenglamaning $y_1(x), y_2(x)$ yechimlari chiziqli erkli bo'lib, uning umumiy yechimi

$$y(x) = A_1 y_1(x) + A_2 y_2(x), \quad A_j = const$$

ko'rinishida yoziladi.

Faraz qilaylik,

$$a_1 y_1(x) + a_2 y_2(x) = 0, \quad a_j = const \quad (3.3.16)$$

bo'lsin. Bu tenglikda $x = x_0$ deb (3.3.12), (3.3.14) boshlang'ich shartlardan foydalansak,

$$a_1 y_1(x_0) + a_2 y_2(x_0) = 0, \quad a_1 + a_2 \cdot 0 = 0, \quad a_1 = 0$$

Endi quyidagi

$$a_1 y_1'(x) + a_2 y_2'(x) = 0$$

tenglikda $x = x_0$ deb

$$a_1 \cdot 0 + a_2 \cdot 1 = 0, \quad a_2 = 0$$

ekanligini topamiz. Demak, (3.3.16) tenglik faqat $a_1 = 0, a_2 = 0$ bo'lganda bajarilar ekan. Shuning uchun $y_1(x), y_2(x)$ funksiyalar (3.3.1) differensial tenglamaning F.Y.S ni tashkil qiladi.

4-§. Eyri tenglamasi

Ushbu

$$y'' + xy = 0 \quad (3.4.1)$$

ko‘rinishdagi tenglamaga Eyri differensial tenglamasi deyiladi. Bu differensial tenglama matematik fizikaning ko‘p sohalarida, shu jumladan Kvant mexanikasining ayrim masalalarini o‘rganishda qo‘llanilib kelmoqda. (3.4.1) tenglama o‘zgaruvchan koeffitsiyentli eng sodda differensial tenglama bo‘lishiga qaramasdan, u elementar funksiyalar yordamida yechilmaydi. Lekin bu differensial tenglamada

$$p(x) = 0, \quad q(x) = x$$

bo‘lgani uchun, bu koeffitsiyentlar eng sodda golomorf funksiyalardir. Shuning uchun (3.4.1) differensial tenglamaning yechimini

$$y(x) = \sum_{k=0}^{\infty} c_k x^k \quad (3.4.2)$$

darajali qator ko‘rinishida izlash mumkin. Avvalo (3.4.2) darajali qatorni formal ravishda ketma-ket ikki marta differensiallab, so‘ngra topilgan y' va y'' -hosilalarni (3.4.1) tenglamaga qo‘ysak

$$\sum_{k=2}^{\infty} k(k-1)c_k x^{k-2} + \sum_{k=0}^{\infty} c_k x^{k+1} = 0, \quad (3.4.3)$$

ya‘ni

$$\sum_{k=0}^{\infty} (k+2)(k+1)c_{k+2} x^k + \sum_{k=1}^{\infty} c_{k-1} x^k = 0$$

munosabat kelib chiqadi. Bu tenglikda x^k ning oldidagi koeffitsiyentlarni nolga tenglashtirib

$$\begin{aligned} c_2 \cdot 2 \cdot 1 &= 0, \\ c_3 \cdot 3 \cdot 2 + c_0 &= 0, \\ \dots\dots\dots \\ c_k \cdot k \cdot (k-1) + c_{k-3} &= 0, \\ c_{k+2} \cdot (k+2) \cdot (k+1) + c_{k-1} &= 0, \\ \dots\dots\dots \end{aligned} \quad (3.4.4)$$

sistemani hosil qilamiz. Bundan

$$c_2 = 0, \quad c_{k+2} = -\frac{1}{(k+1)(k+2)} c_{k-1}, \quad k \geq 1,$$

ya‘ni

$$c_k = -\frac{1}{k(k+1)} c_{k-3}, \quad k \geq 3,$$

munosabatlarni topamiz. Ko‘rinib turibdiki

$$\begin{aligned}
1) \quad c_{3m} &= \frac{(-1)^m}{3m(3m-1)\dots 3 \cdot 2} c_0, \\
2) \quad c_{3m+1} &= \frac{(-1)^m}{(3m+1)3m\dots 4 \cdot 3} c_1, \\
3) \quad c_{3m+2} &= \frac{(-1)^m}{(3m+2)(3m+1)3m\dots 5 \cdot 4} c_2 = 0.
\end{aligned}$$

Aytaylik, $c_0 = 1, c_1 = 0$ bo'lsin. U holda

$$y_1(0) = 1, y_1'(0) = 0$$

boshlang'ich shartlarni qanoatlantiruvchi (3.4.1) differensial tenglamaning yechimi

$$y_1(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{3m}}{3m(3m-1)\dots 3 \cdot 2} \quad (3.4.5)$$

ko'rinishdagi darajali qatordan iborat bo'ladi.

Faraz qilaylik $c_0 = 0, c_1 = 1$ bo'lsin. U holda (3.4.1) differensial tenglamaning

$$y_2(0) = 0, y_2'(0) = 1$$

boshlang'ich shartlarni qanoatlantiruvchi yechimi

$$y_2(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{3m+1}}{(3m+1)3m(3m-1)\dots 4 \cdot 3} \quad (3.4.6)$$

ko'rinishdagi darajali qatordan iborat bo'ladi.

3.4.1-teorema. (3.4.5) va (3.4.6) darajali qatorlar $\forall x \in (-\infty, \infty) = R$ - haqiqiy sonlar o'qida yaqinlashuvchi bo'lib, ular yordamida aniqlangan $y_1(x), y_2(x)$ funksiyalar (3.4.1) differensial tenglamaning F.Y.S ni tashkil qiladi.

Isbot. Avvalo (3.4.5) darajali qatorning yaqinlashish radiusini ushbu

$$R = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right|$$

formuladan foydalanib hisoblaymiz:

$$\begin{aligned}
c_{3m} &= \frac{(-1)^m}{3m(3m-1)\dots 3 \cdot 2} c_0, \\
c_{3m+3} &= \frac{(-1)^{m+1}}{(3m+3)(3m+2)(3m+1)\dots 3 \cdot 2} c_0
\end{aligned}$$

$$\begin{aligned}
R &= \lim_{m \rightarrow \infty} \left| \frac{c_{3m}}{c_{3m+3}} \right| = \lim_{m \rightarrow \infty} \left| \frac{(-1)^m}{3m(3m-1)\dots 3 \cdot 2} \cdot \frac{(3m+3)(3m+2)\dots 3 \cdot 2}{(-1)^{m+1}} \right| = \\
&= \lim_{m \rightarrow \infty} \frac{(3m+3)(3m+2)(3m+1)}{1} = \infty.
\end{aligned}$$

Demak, (3.4.5) darajali qatorning yaqinlashish radiusi $R = \infty$ bo'lgani uchun u $\forall x \in (-\infty, \infty)$ oraliqda yaqinlashadi. Shunday qilib, (3.4.5) tenglik yordamida aniqlangan $y_1(x)$ funksiya barcha tartibli hosilalariga ega bo'ladi. Bundan tashqari (3.4.7) formula bilan aniqlangan $y_1(x)$ funksiya (3.4.1) differensial tenglamani qanoatlantirishini ko'rsatish qiyinchilik tug'dirmaydi. Xuddi shuningdek, (3.4.6) darajali qatorning $\forall x \in (-\infty, \infty)$ oraliqda yaqinlashuvchi ekanligini hamda $y_2(x)$ funksiya (3.4.1) differensial tenglamaning yechimi ekanligini ham ko'rsatish mumkin.

Endi $y_1(x)$ va $y_2(x)$ yechimlarning chiziqli erkliligini ko'rsatamiz. Buning uchun teskarisini faraz qilamiz, ya'ni $y_1(x)$ va $y_2(x)$ yechimlar chiziqli bog'langan bo'lsin. U holda ushbu

$$C_1 y_1(x) + C_2 y_2(x) = 0, \quad C_1^2 + C_2^2 \neq 0 \quad (3.4.7)$$

tenglikni qaraymiz. Bunda $x = 0$ bo'lsin. U holda $y_2(0) = 0$, $y_1(0) = -\frac{1}{3 \cdot 2}$

bo'lgani uchun

$$-C_1 \cdot \frac{1}{3 \cdot 2} = 0 \quad C_1 = 0$$

bo'lishini topamiz. U holda (3.4.7) tenglik

$$C_2 y_2(x) = 0$$

ko'rinishni oladi. Oxirgi tenglikni differensiallab

$$C_2 y_2'(x) = 0$$

hosil qilamiz va bunda $x = 0$ deb

$$C_2 y_2'(0) = 0, \quad C_2 \left(-\frac{1}{4 \cdot 3} \right) = 0, \quad C_2 = 0$$

ekanini topamiz. Demak, (3.4.7) tenglik faqat $C_1 = 0$, $C_2 = 0$ bo'lganda bajarilar ekan. Shuning uchun $y_1(x)$ va $y_2(x)$ funksiyalar (3.4.1) differensial tenglamaning F.Y.S ni tashkil qilar ekan. Endi $y_1(x)$ va $y_2(x)$ funksiyalardan tuzilgan Vronskiy determinantini hisoblaymiz:

$$W(x) = W\{y_1(x), y_2(x)\}|_{x=0} = W(0) = 1$$

Demak, (3.4.1) differensial tenglamaning umumiy yechimini

$$y(x) = C_1 y_1 + C_2 y_2$$

ko'rinishda yozish mumkin ekan. Bu yerda C_j - ixtiyoriy o'zgarmaslar.

Mustaqil yechish uchun mashqlar [8], §18, №724-738; [21], §13, №700-716.

5-§. Bessel tenglamasi

Ushbu

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0 \quad (3.5.1)$$

ko‘rinishdagi tenglamaga Bessel tenglamasi deyiladi. Bu yerda ν -haqiqiy parametr. $x=0$ nuqta Bessel tenglamasi uchun regulyar maxsus nuqta bo‘ladi. Bessel tenglamasi matematika va nazariy fizikaning bir qancha masalalarida uchraydi. Bu paragrafda Bessel tenglamasining umumiy yechimini qurish algoritmini bayon qilamiz.

Bessel tenglamasining yechimini

$$y(x) = x^\alpha \sum_{k=0}^{\infty} c_k x^k, \quad c_0 \neq 0 \quad (3.5.2)$$

ko‘rinishda izlaymiz. Avvalo bu darajali qatorni formal ikki marta differensiallab (3.5.1) tenglamaga qo‘yib α va c_k larning qiymatlarini topamiz:

$$\sum_{k=0}^{\infty} c_k (k + \alpha)(k + \alpha - 1)x^{k+\alpha} + \sum_{k=0}^{\infty} c_k (k + \alpha)x^{k+\alpha} + (x^2 - \nu^2) \sum_{k=0}^{\infty} c_k x^{k+\alpha} = 0.$$

Bu tenglikda x ning darajalari oldidagi koeffitsiyentlarni nolga tenglashtirib α va c_j larga nisbatan tenglamalar sistemasini hosil qilamiz:

$$\begin{aligned} (\alpha^2 - \nu^2)c_0 &= 0, \\ ((\alpha + 1)^2 - \nu^2)c_1 &= 0, \\ [(\alpha + 2)^2 - \nu^2]c_2 + c_0 &= 0, \\ [(\alpha + 3)^2 - \nu^2]c_3 + c_1 &= 0, \\ &\dots\dots\dots \\ [(\alpha + k)^2 - \nu^2]c_k + c_{k-2} &= 0, \\ &\dots\dots\dots \end{aligned}$$

Berilishiga ko‘ra $c_0 \neq 0$ bo‘lgani uchun, birinchi tenglamadan $\alpha = \pm \nu$ ekanligini topamiz. Avvalo $\alpha = \nu \geq 0$ holni qaraymiz. Bu holda ikkinchi tenglamadan $c_1 = 0$ va $c_{2p-1} = 0, \forall p = \overline{1, n}$ hamda

$$c_{2p} = \frac{(-1)^p c_0}{2^{2p} (\nu + 1)(\nu + 2)\dots(\nu + p)p!}, \quad p = 1, 2, \dots$$

munosabatlarni topamiz. Bu yerda c_0 -ixtiyoriy son bo‘lgani uchun, c_{2p} ni soddalashtirish maqsadida $c_0 = \frac{1}{2^\nu \Gamma(\nu + 1)}$ deb olish maqsadga muvofiq

bo‘ladi. Bunda $\Gamma(p + 1)$ -Eylerning “gamma” funksiyasi. Ushbu

$$\Gamma(p + 1) = p!, \quad \Gamma(\nu + p + 1) = (\nu + 1)(\nu + 2)\dots(\nu + p)\Gamma(\nu + 1)$$

munosabatlardan foydalanib

$$c_{2p} = \frac{(-1)^p}{2^{2p+\nu} \Gamma(p+1) \Gamma(\nu+p+1)}, \quad p=1,2,\dots$$

tasvirni hosil qilamiz.

Dalamber belgisidan foydalanib va $c_{2p-1} = 0$ ekanligini inobatga olsak (3.5.2) darajali qatorning $\nu \geq 0, \forall x \in R$ da yaqinlashuvchi ekanligini ko'rsatish mumkin. Bundan tashqari (3.5.2) formula bilan aniqlangan $y(x)$ funksiyani (3.5.1) differensial tenglamani yechimidan iborat ekanligini ham ko'rsatish mumkin. Bu yechimga ν tartibli 1-tur Bessel funksiyasi deyiladi va $J_\nu(x)$ kabi belgilanadi, ya'ni

$$J_\nu(x) = \sum_{p=0}^{\infty} \frac{(-1)^p \left(\frac{x}{2}\right)^{2p+\nu}}{\Gamma(p+1) \Gamma(\nu+p+1)}, \quad \nu \geq 0. \quad (3.5.3)$$

Agar $\alpha = -\nu < 0$ bo'lib, butun son bo'lmasa, u holda xuddi shuningdek (3.5.1) differensial tenglamaning yana bir (3.5.2) ko'rinishdagi yechimini qurish mumkin. Bu esa (3.5.3) tenglikda ν ni $-\nu$ ga almashtirish orqali aniqlanadi:

$$J_{-\nu}(x) = \sum_{p=0}^{\infty} \frac{(-1)^p \left(\frac{x}{2}\right)^{2p-\nu}}{\Gamma(p+1) \Gamma(-\nu+p+1)}, \quad \nu > 0 \quad (3.5.4)$$

Odatda $J_{-\nu}(x)$ -funksiyani ν -tartibli 1-tur Bessel funksiyasi deyiladi.

Agar $\nu > 0$ bo'lib, butun son bo'lmasa u holda, $J_\nu(x)$ va $J_{-\nu}(x)$ funksiyalar chiziqli erkli bo'ladi, ya'ni

$$c_1 J_\nu(x) + c_2 J_{-\nu}(x) = 0, \quad x \in R$$

tenglik faqat $c_1 = c_2 = 0$ bo'lganda bajariladi. Shuning uchun $\nu > 0$ bo'lib, butun son bo'lmasa, $J_\nu(x)$ va $J_{-\nu}(x)$ funksiyalar (3.5.1) differensial tenglamaning F.Y.S ni tashkil qiladi hamda ushbu

$$y_\nu(x) = c_1 J_\nu(x) + c_2 J_{-\nu}(x), \quad c_j = const$$

funksiya (3.5.1) Bessel tenglamasining umumiy yechimi bo'ladi.

Agar ν butun, ya'ni $\nu = n$ ko'rinishdagi son bo'lsa, u holda (3.5.4) formula yaroqsiz. Chunki, uning dastlabki n ta hadining maxraji cheksizga aylanishi mumkin. Ammo (3.5.4) formulada $\nu \rightarrow n$ da limitga o'tish mumkin:

$$\frac{1}{\Gamma(-n+p+1)} = 0, \quad p = \overline{0, n-1}.$$

Har bir tayinlangan x larda

$$J_{-n}(x) = \lim_{\nu \rightarrow n} J_{-\nu}(x) = \sum_{p=0}^{\infty} \frac{(-1)^p \left(\frac{x}{2}\right)^{2p-\gamma_n}}{\Gamma(p+1) \Gamma(-\gamma_n+p+1)},$$

munosabat o'rinli. Bunda $p = n + m$ deb

$$J_{-n}(x) = (-1)^n \sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma(m+1)\Gamma(n+m+1)} \left(\frac{x}{2}\right)^{2m+n} = (-1)^n J_n(x)$$

tenglikka ega bo‘lamiz. Bu tenglik $J_n(x)$ va $J_{-n}(x)$ funksiyalarning n ning butun qiymatlarida chiziqli bog‘liq ekanligini ko‘rsatadi.

Butun n larda (3.5.1) tenglamaning ikkinchi chiziqli erkli yechimiga 2-tur n - tartibli Bessel funksiyasi deyiladi va $Y_n(x)$ orqali belgilanadi. $Y_n(x)$ funksiyani Ostragradiskiy – Liuvill formulasidan foydalanib ham topish mumkin. Bunga to‘xtalib o‘tirmasdan $x \rightarrow +0$ da $Y_n(x)$ funksiya quyidagi

$$Y_n(x) = \begin{cases} c_n x^{-n} [1 + o(1)], & n \in N \\ c_0 \ln x [1 + o(1)], & n = 0 \end{cases}$$

asimptotikalarni qanoatlantirishini e’tirof etamiz.

Shunday qilib butun ν larda (3.5.1) differensial tenglamaning umumiy yechimi

$$y_\nu(x) = c_1 J_\nu(x) + c_2 Y_\nu(x)$$

ko‘rinishda bo‘lar ekan.

Mustaqil yechish uchun mashqlar [8], §18, №739-741, 744-751.

Keyinchalik A -matritsani $A: R^n \rightarrow R^n$, A -chiziqli akslantirishning R^n fazoning e_1, e_2, \dots, e_n bazisdagi matritsasi deb hisoblaymiz. Berilgan (4.1.3) sistema trivial $y(x) \equiv 0$ yechimga ega ekanligi ravshan. Biz (4.1.3) sistemaning trivial bo'lmagan yechimini

$$y(x) = e^{\lambda x} h \quad (4.1.4)$$

ko'rinishida izlaymiz, bu yerda $h \neq 0$ sonli vektor: $h = (h_1, h_2, \dots, h_n)^T$.

Bu (4.1.4) tenglik orqali aniqlangan $y(x)$ vektor funksiyani (4.1.3) sistemaga qo'yib, $\lambda e^{\lambda x} h = A e^{\lambda x} h$, ya'ni

$$Ah = \lambda h, \quad h \neq 0$$

chiziqli algebraik tenglamalar sistemasiga ega bo'lamiz. Shuning uchun chiziqli algebraning ayrim tushunchalarini bayon qilamiz.

4.1.1-ta'rif. Noldan farqli $h \in R^n$ vektor uchun ushbu

$$Ah = \lambda h \quad (4.1.5)$$

munosabato'rinli bo'lsa, λ -songa A matritsaning xos qiymati, h -vektorga esa uning xos vektori deyiladi.

Ma'lumki, A matritsaning xos qiymatlari ushbu

$$\det(A - \lambda I) = 0 \quad (4.1.6)$$

xarakteristik tenglamadan topiladi. Bu yerda I -birlik matritsa.

4.1.1-teorema. Agar A matritsaning $\lambda_1, \lambda_2, \dots, \lambda_n$ xos qiymatlari har xil bo'lsa, u holda uning h_1, h_2, \dots, h_n xos vektorlari chiziqli erkli bo'lib, R^n fazoning bazisini tashkil qiladi.

Bu teoremani quyidagicha ham bayon qilish mumkin.

4.1.2-teorema. Agar A -matritsaning $\lambda_1, \lambda_2, \dots, \lambda_n$ xos qiymatlari har xil bo'lsa, u holda shunday $n \times n$ -o'lchamli xosmas T ($\det T \neq 0$) matritsa mavjud bo'lib, quyidagi

$$T^{-1}AT = \Lambda \equiv \begin{vmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \lambda_n \end{vmatrix} \quad (4.1.7)$$

munosabato'rinli bo'ladi. Bu yerda

$$T = (h_1, h_2, \dots, h_n)$$

bo'lib, h_j - A matritsaning λ_j xos qiymatlariga mos keluvchi xos vektorlar.

4.1.2-lemma. Ushbu $y(x) = e^{\lambda x} h$ vektor-funksiya (4.1.3) bir jinsli sistemaning yechimi bo'lishi uchun λ soni A matritsaning xos qiymati, h esa uning xos vektori bo'lishi zarur va yetarli.

4.1.3-teorema. Agar A matritsaning $\lambda_1, \lambda_2, \dots, \lambda_n$ xos qiymatlari har xil bo'lsa, u holda:

1) Ushbu

$$y(x) = c_1 e^{\lambda_1 x} h_1 + c_2 e^{\lambda_2 x} h_2 + \dots + c_n e^{\lambda_n x} h_n \quad (4.1.8)$$

vektor-funksiya (4.1.3) differensial tenglamalar sistemasining yechimidan iborat bo'ladi. Bu yerda c_j - ixtiyoriy haqiqiy o'zgaruvchilar.

2) Agar $y(x)$ -vektor-funksiya (4.1.3) sistemaning biror yechimi bo'lsa, u holda c_1, c_2, \dots, c_n -o'zgaruvchilarning shunday qiymatlari topiladiki $y(x)$ (4.1.8) ko'rinishda bo'ladi.

Isbot. Teorema birinchi bandining isboti 4.1.1-lemma va 4.1.2-lemmadan kelib chiqadi. Shuning uchun teoremaning ikkinchi bandini isbotlaymiz. Buning uchun quyidagi

$$y(x) = Tz(x) \quad (4.1.9)$$

almashtirishdan foydalanib, (4.1.3) sistemani

$$T \frac{dz(x)}{dx} = ATz(x)$$

ko'rinishda yozib olamiz. Bu tenglikning ikkala tomonini T^{-1} -teskari matritsaga chap tomondan ko'paytirib,

$$\frac{dz(x)}{dx} = T^{-1}ATz(x),$$

ya'ni

$$\frac{dz(x)}{dx} = \Lambda z(x) \quad (4.1.10)$$

tenglikni hosil qilamiz. (4.1.10) differensial tenglamani koordinatalarda yozsak,

$$\frac{dz_1(x)}{dx} = \lambda_1 z_1(x), \frac{dz_2(x)}{dx} = \lambda_2 z_2(x), \dots, \frac{dz_n(x)}{dx} = \lambda_n z_n(x) \quad (4.1.11)$$

sodda differensial tenglamalar hosil bo'ladi. Bu tenglamalarni birin-ketin yechib,

$$z_1(x) = c_1 e^{\lambda_1 x}, z_2(x) = c_2 e^{\lambda_2 x}, \dots, z_n(x) = c_n e^{\lambda_n x} \quad (4.1.12)$$

funksiyalarni topamiz.

Agar R^n fazoning birlik vektorlarini

$$f_1 = (1, 0, 0, \dots, 0), f_2 = (0, 1, 0, \dots, 0), \dots, f_n = (0, 0, 0, \dots, 1)$$

ko'rinishda belgilab olsak, u holda $z(x) = (z_1, z_2, \dots, z_n)^T$ vektor-funksiyani ushbu

$$z(x) = c_1 e^{\lambda_1 x} f_1 + c_2 e^{\lambda_2 x} f_2 + \dots + c_n e^{\lambda_n x} f_n \quad (4.1.13)$$

ko'rinishda yozish mumkin. Xosmas T matritsaning ustunlari A matritsaning h_j xos vektorlaridan iborat bo'lgani uchun

$$Tf_k = h_k \quad (4.1.14)$$

munosabat bajariladi. (4.1.13) tenglikning ikkala tomoniga chapdan T matritsani ta'sir qildirish natijasida (4.1.3) differensial tenglamalar sistemasining yechimini hosil qilamiz:

$$\begin{aligned} y(x) &= Tz(x) = c_1 e^{\lambda_1 x} T f_1 + \dots + c_n e^{\lambda_n x} T f_n = \\ &= c_1 e^{\lambda_1 x} h_1 + c_2 e^{\lambda_2 x} h_2 + \dots + c_n e^{\lambda_n x} h_n. \end{aligned}$$

Teoremaning bu bandini boshqacha ham isbotlash mumkin. Haqiqatan ham h_1, h_2, \dots, h_n vektorlar R^n fazoda bazis tashkil qilgani uchun ushbu

$$y(x) = a_1(x)h_1 + a_2(x)h_2 + \dots + a_n(x)h_n \quad (4.1.15)$$

yoyilma $\forall x \in R = (-\infty, \infty)$ da o'rinli bo'ladi. Bu $y(x)$ -vektor-funksiyani (4.1.3) sistemaga qo'yib

$$\begin{aligned} a'_1(x)h_1 + a'_2(x)h_2 + \dots + a'_n(x)h_n &= a_1(x)Ah_1 + a_2(x)Ah_2 + \dots + a_n(x)Ah_n = \\ &= \lambda_1 a_1(x)h_1 + \lambda_2 a_2(x)h_2 + \dots + \lambda_n a_n(x)h_n \end{aligned}$$

munosabatni hosil qilamiz. h_1, h_2, \dots, h_n vektorlarning chiziqli erkliligini e'tiborga olsak, oxirgi tenglikdan quyidagi

$$a'_1(x) = \lambda_1 a_1(x), \quad a'_2(x) = \lambda_2 a_2(x), \dots, \quad a'_n(x) = \lambda_n a_n(x)$$

sodda differensial tenglamalar kelib chiqadi. Ularning har birini ketma-ket yechib,

$$a_1(x) = c_1 e^{\lambda_1 x}, \quad a_2(x) = c_2 e^{\lambda_2 x}, \dots, \quad a_n(x) = c_n e^{\lambda_n x}$$

funksiyalarni topib olamiz. Topilgan bu funksiyalarni (4.1.15) tenglikka qo'yib,

$$y(x) = c_1 e^{\lambda_1 x} h_1 + c_2 e^{\lambda_2 x} h_2 + \dots + c_n e^{\lambda_n x} h_n$$

(4.1.3) sistemaning yechimini hosil qilamiz.

Aytaylik, A matritsaning barcha elementlari haqiqiy sonlardan iborat bo'lsin. Bu holda (4.1.3) differensial tenglamalar sistemasining haqiqiy yechimlarini topish algoritmini bayon qilamiz.

4.1.3-lemma. Agar A -haqiqiy elementli matritsa bo'lib, λ -soni uning xos qiymati, h -esa uning xos vektori bo'lsa, u holda $\bar{\lambda}$ -xos qiymatga, \bar{h} -xos vektor mos keladi.

Isbot. Shartga ko'ra

$$Ah = \lambda h, \quad h \neq 0.$$

Bu tenglikda qatnashayotgan hadlarning kompleks qo'shmasigao'tib,

$$\overline{Ah} = \overline{\lambda h}$$

munosabatni olamiz. A -haqiqiy elementli matritsa bo'lgani uchun

$$\overline{A} = (\overline{a_{ij}})_{i,j=1,\overline{n}} = (a_{ij})_{i,j=1,\overline{n}} = A$$

o'rinli bo'ladi. Bunga asosan yuqoridagi tenglik ushbu

$$\overline{Ah} = \overline{\lambda h}$$

ko'rinishni oladi. Bundan o'z navbatida $\bar{\lambda}$ soni A matritsaning xos qiymati, \bar{h} -esa uning xos vekroti ekanligi kelib chiqadi.

Agar λ xos qiymat haqiqiy son, ya'ni $\bar{\lambda} = \lambda$ bo'lsa, u holda h xos vektorni haqiqiy qilib tanlash mumkin. Shunday qilib biz

$$y(x) = e^{\lambda x} h$$

haqiqiy yechimga ega bo'lamiz.

Agar λ xos qiymat kompleks son bo'lsa, u holda

$$y(x) = e^{\lambda x} h$$

ham kompleks bo'ladi. Bu yechimlarning haqiqiy va mavhum qismlari

$$y_1(x) = \operatorname{Re}\{e^{\lambda x} h\}, y_2(x) = \operatorname{Im}\{e^{\lambda x} h\}$$

ham (4.1.3) differensial tenglamalar sistemasining yechimlaridan iborat bo'ladi. Bundan tashqari (4.1.3) sistema kompleks qo'shma

$$\bar{y} = e^{\bar{\lambda} x} \bar{h}$$

yechimga ham ega bo'ladi. Bu yechimning haqiqiy va mavhum qismlari

$$\operatorname{Re}\{e^{\bar{\lambda} x} \bar{h}\} = y_1(x), \operatorname{Im}\{e^{\bar{\lambda} x} \bar{h}\} = -y_2(x)$$

ko'rinishda bo'lib, (4.1.3) sistemaning haqiqiy yechimlar juftligini beradi.

Berilgan (4.1.3) differensial tenglamalar sistemasining haqiqiy yechimlari quyidagicha topiladi. Buning uchun A matritsaning barcha haqiqiy xos qiymatlarini $\lambda_1, \lambda_2, \dots, \lambda_k$ orqali, kompleks xos qiymatlarini $\lambda_{k+1}, \overline{\lambda_{k+1}}, \lambda_{k+2}, \overline{\lambda_{k+2}}, \dots$ orqali belgilab olamiz. U holda (4.1.3) differensial tenglamalar sistemasining ixtiyoriy haqiqiy yechimi

$$y(x) = c_1 e^{\lambda_1 x} h_1 + c_2 e^{\lambda_2 x} h_2 + \dots + c_k e^{\lambda_k x} h_k + c_{k+1} \operatorname{Re}\{e^{\lambda_{k+1} x} h_{k+1}\} + c_{k+2} \operatorname{Im}\{e^{\lambda_{k+1} x} h_{k+1}\} + \dots \quad (4.1.16)$$

ko'rinishda bo'ladi.

Shunday qilib, A matritsa har xil xos qiymatlarga (ya'ni oddiy) ega bo'lgan holda, (4.1.3) differensial tenglamalar sistemaning barcha yechimlarini topish algoritmi quyidagi bosqichlardan iborat bo'ladi:

1. A matritsaning barcha oddiy xos qiymatlari ushbu

$$\Delta(\lambda) = \det(A - \lambda I) = 0$$

xarakteristik tenglamadan topiladi.

2. Ushbu

$$A h_j = \lambda_j h_j$$

algebraik tenglamalar sistemasini yechib, h_1, h_2, \dots, h_n - xos vektorlar topiladi.

3. Nihoyat, (4.1.16) formula bo'yicha (4.1.3) differensial tenglamalar sistemasi yechimi topiladi.

2-§. O'zgarmas koeffitsiyentli chiziqli bir jinsli differensial tenglamalar sistemasini karrali xos qiymatlar holda yechish algoritmi

1. Aytaylik R^n fazoda berilgan simmetrik bo'lmagan A chiziqli operatorning biror $\{e_1, e_2, \dots, e_n\}$ bazisdagi matritsasi ushbu

$$A = (a_{jk}), \quad j, k = \overline{1, n}$$

ko‘rinishda bo‘lib, u k karrali λ xos qiymatga ega bo‘lsin. U holda uning chiziqli erkli xos vektorlari soni k dan kam bo‘ladi. Shuning uchun A matritsaning xos vektorlari R^n fazoning bazisini tashkil qilmaydi. Ammo R^n fazosining Jordan bazisini tuzish mumkin. Buning uchun Jordan katagi va zanjiri hamda yopishgan (yoki ergashgan) vektor tushunchalarini bayon qilish zarur.

4.2.1-ta’rif. Ushbu

$$J = \begin{vmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \lambda \end{vmatrix}$$

ko‘rinishdagi matritsaga Jordan katagi deyiladi. Bu matritsa k karrali yagona λ xos qiymatga ega bo‘lib, k uning o‘lchamini bildiradi.

J matritsaning λ xos qiymatiga mos keluvchi barcha xos vektorlarini topaylik. Buning uchun ushbu

$$Jx = \lambda x, \quad x = (x_1, x_2, \dots, x_k)^T$$

tenglamaning nolmas yechimlarini topamiz. Bu tenglamani koordinatalarda yozib, quyidagi

$$\lambda x_1 + x_2 = \lambda x_1, \lambda x_2 + x_3 = \lambda x_2, \dots, \lambda x_{k-1} + x_k = \lambda x_{k-1}, \lambda x_k = \lambda x_k$$

tenglamalarni hosil qilamiz. Bunda $x_2 = 0, x_3 = 0, \dots, x_k = 0$ bo‘lib, x_1 - ixtiyoriy son. Shuning uchun J matritsa yagona

$$f_1 = (1, 0, 0, \dots, 0)^T$$

xos vektorga ega bo‘ladi. R^n fazoning bundan boshqa

$$f_2 = (0, 1, 0, \dots, 0)^T, f_3 = (0, 0, 1, 0, \dots, 0)^T, \dots, f_k = (0, 0, 0, \dots, 1)^T$$

birlik vektorlariga J matritsa quyidagicha ta’sir qiladi:

$$Jf_1 = \lambda f_1, Jf_2 = \lambda f_2 + f_1, \dots, Jf_k = \lambda f_k + f_{k-1}.$$

Aytaylik, λ soni A matritsaning k karrali xos qiymati bo‘lsin.

4.2.2-ta’rif. Ushbu

$$Ah_1 = \lambda h_1, \quad h_1 \neq 0,$$

$$Ah_2 = \lambda h_2 + h_1,$$

$$Ah_3 = \lambda h_3 + h_2,$$

$$\cdots \cdots \cdots$$

$$Ah_k = \lambda h_k + h_{k-1}$$

tengliklarni qanoatlantiruvchi $\{h_1, h_2, \dots, h_k\}$ vektorlar sistemasiga A matritsaning Jordan zanjiri deyiladi. Bunda h_1 - A -matritsaning xos vektori, h_2, h_3, \dots, h_k vektorlarga esa unga yopishgan vektorlar deyiladi.

Demak, yuqoridagi f_1, f_2, \dots, f_k vektorlar J Jordan katagining Jordan zanjirini tashkil qiladi.

4.2.1-teorema (Jordan). Ixtiyoriy A matritsaning barcha xos qiymatlariga mos keluvchi Jordan zanjiridan tuzilgan vektorlar sistemasi R^n fazoning bazisini tashkil qiladi.

Endi A matritsaning Jordan bazisidagi ko‘rinishini ifodalovchi tasdiqni keltiramiz.

4.2.2-teorema. Ixtiyoriy A matritsa uchun, shunday xosmas $T(\det T \neq 0)$ matritsa topilib, quyidagi

$$T^{-1}AT = \begin{vmatrix} J_1 & \dots & 0 & 0 \\ 0 & J_2 & \dots & 0 \\ & & \ddots & \\ 0 & 0 & \dots & J_j \end{vmatrix}$$

tengliko‘rinli bo‘ladi. Bu yerda J_1, J_2, \dots, J_j matritsalar mos ravishda k_1, k_2, \dots, k_j o‘lchamli Jordan kataklardir:

$$J_k = \begin{vmatrix} \lambda_k & 1 & 0 & \dots & 0 & 0 \\ 0 & \lambda_k & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \lambda_k & 1 \\ 0 & 0 & 0 & \dots & 0 & \lambda_k \end{vmatrix}.$$

Bunda $k_1 + k_2 + \dots + k_n = n$ bo‘lib, $\{\lambda_j\}_{j=1}^n - A$ matritsaning $k_j, j = \overline{1, n}$ karrali xos qiymatidir.

2. Quyidagi n noma‘lumli n ta differensial tenglamalar sistemasini qaraylik:

$$\frac{dy}{dx} = Ay, \quad y(x) = (y_1(x), y_2(x), \dots, y_n(x))^T, \quad (4.2.1)$$

$$x \in R, \quad A = (a_{j,k}), \quad a_{j,k} = \text{const}, \quad j, k = \overline{1, n}.$$

Bu yerda A -o‘zgarmas simmetrik bo‘lmagan matritsa, $y(x)$ -noma‘lum vektor-funksiya. Faraz qilaylik, λ soni A matritsaning k karrali xos qiymati bo‘lib, $h_1, h_2, h_3, \dots, h_k$ vektorlar uning Jordan zanjiridan iborat bo‘lsin. U holda bu Jordan zanjiriga (4.2.1) differensial tenglamalar sistemasining k ta yechimi mos kelishini ko‘rsatamiz. Buning uchun, avvalo ushbu

$$\frac{dz}{dx} = Jz, \quad z(x) = (z_1(x), \dots, z_k(x))^T \quad (4.2.2)$$

differensial tenglamalar sistemasining yechimini topamiz. Bunda J -Jordan katagidan iborat. (4.2.2) sistemani koordinatalarda yozamiz:

$$z_1' = \lambda z_1 + z_2,$$

$$z'_2 = \lambda z_2 + z_3,$$

.....

$$z'_{k-1} = \lambda z_{k-1} + z_k,$$

$$z'_k = \lambda z_k.$$

So'ngra bu differensial tenglamalarni oxirgisidan boshlab ketma-ket yechamiz:

$$z_k(x) = c_k e^{\lambda x},$$

$$z_{k-1}(x) = c_{k-1} e^{\lambda x} + c_k x e^{\lambda x} = \left(c_{k-1} + c_k \frac{x}{1!} \right) e^{\lambda x},$$

.....

$$z_1(x) = \left(c_1 + c_2 \frac{x}{1!} + \dots + c_k \frac{x^{k-1}}{(k-1)!} \right) e^{\lambda x}.$$

Demak, (4.2.2) differensial tenglamalar sistemasining ixtiyoriy yechimini

$$z(x) = c_1 z^{(1)}(x) + c_2 z^{(2)}(x) + \dots + c_k z^{(k)}(x)$$

ko'rinishda yozish mumkin ekan. Bu yerda

$$z^{(1)}(x) = e^{\lambda x} f_1,$$

$$z^{(2)}(x) = e^{\lambda x} \left(f_2 + \frac{x}{1!} f_1 \right),$$

.....

$$z^{(k)}(x) = e^{\lambda x} \left(f_k + \frac{x}{1!} f_{k-1} + \dots + \frac{x^{k-1}}{(k-1)!} f_1 \right)$$

bo'lib, f_1, f_2, \dots, f_k vektorlar J matritsaning Jordan zanjiridan iborat.

Quyidagi

$$P_1(x) = f_1,$$

$$P_2(x) = f_2 + \frac{x}{1!} f_1,$$

.....

$$P_k(x) = f_k + \frac{x}{1!} f_{k-1} + \dots + \frac{x^{k-1}}{(k-1)!} f_1$$

belgilashdan foydalansak, yuqoridagi tengliklarni

$$z^{(1)}(x) = e^{\lambda x} P_1(x), \quad z^{(2)}(x) = e^{\lambda x} P_2(x),$$

$$z^{(3)}(x) = e^{\lambda x} P_3(x), \dots, z^{(k)}(x) = e^{\lambda x} P_k(x) \quad (4.2.3)$$

ko'rinishida yozish mumkin. $P_j(x)$, $j = \overline{1, k}$ vektor-funksiyalar ushbu

$$\frac{dP_j(x)}{dx} = P_{j-1}(x), \quad j = \overline{1, k}$$

tenglikni qanoatlantiradi. $\{f_1, f_2, \dots, f_k\}$ -vektorlar-sistemi J matritsaning Jordan zanjirini tashkil qilgani uchun

munosabato‘rinli. Yuqoridagi h_1, h_2, \dots, h_k vektorlar Jordan zanjirini tashkil qilgani uchun, ushbu

$$AQ_j(x) = \lambda Q_j(x) + Q_{j-1}(x)$$

tenglik bajariladi. Bundan foydalanib quyidagi ayirmani hisoblaymiz:

$$\begin{aligned} y'_j(x) - Ay_j(x) &= \lambda e^{\lambda x} Q_j(x) + e^{\lambda x} Q'_j(x) - \\ - e^{\lambda x} AQ_j(x) &= \lambda e^{\lambda x} Q_j(x) + \lambda e^{\lambda x} Q_{j-1}(x) - \\ - \lambda e^{\lambda x} Q_j(x) - e^{\lambda x} Q_{j-1}(x) &= 0. \end{aligned}$$

Demak, $y_j(x) = e^{\lambda x} Q_j(x)$, $j = \overline{1, k}$ vektor-funksiyalar (4.2.1) differensial tenglamalar sistemasining yechimi bo‘lar ekan.

4.2.3-teorema. Aytaylik, R^n fazoning Jordan bazasi A matritsaning k_j karrali λ_j xos qiymatiga mos keluvchi m ta $h_1^{(j)}, h_2^{(j)}, \dots, h_{k_j}^{(j)}$, $1 \leq j \leq m$, $k_1 + k_2 + \dots + k_m = n$ Jordan zanjiridan iborat bo‘lsin. U holda ushbu

$$y(x) = \sum_{j=1}^m e^{\lambda_j x} \left[C_1^{(j)} Q_1^{(j)}(x) + C_2^{(j)} Q_2^{(j)}(x) + \dots + C_{k_j}^{(j)} Q_{k_j}^{(j)}(x) \right] \quad (4.2.5)$$

vektor-funksiya (4.2.1) differensial tenglamalar sistemasining yechimidan iborat bo‘ladi. Bu yerda $C_1^{(j)}, C_2^{(j)}, \dots, C_{k_j}^{(j)}$, $j = \overline{1, m}$ - ixtiyoriy haqiqiy o‘zgarmlar.

2) Agar $y(x)$ vektor-funksiya (4.2.1) differensial tenglamalar sistemasining biror yechimi bo‘lsa, u holda shunday $C_1^{(j)}, C_2^{(j)}, \dots, C_{k_j}^{(j)}$ haqiqiy o‘zgarmlar sonlar topilib, bu $y(x)$ yechim uchun (4.2.5) tasviro‘rinli bo‘ladi.

Isbot. Teorema birinchi bandini isboti 4.2.1-lemma va superpozitsiya prinspidan kelib chiqadi. Shuning uchun teorema ikkinchi bandining isbotini keltiramiz.

Faraz qilaylik, $y(x)$ vektor-funksiya (4.2.1) differensial tenglamalar sistemasining biror yechimi bo‘lsin. Har bir $x \in R$ larda $y(x)$ yechimni R^n fazoning Jordan bazasi orqali yoyish mumkin:

$$y(x) = \sum_{j=1}^m \left[\xi_1^{(j)}(x) h_1^{(j)} + \xi_2^{(j)}(x) h_2^{(j)} + \dots + \xi_{k_j}^{(j)}(x) h_{k_j}^{(j)} \right]. \quad (4.2.6)$$

Bu $y(x)$ vektor-funksiyaning (4.2.6) ifodasini (4.2.1) differensial tenglamalar sistemasiga qo‘yib Jordan zanjirining ta’rifidan foydalansak,

$$\begin{aligned} \sum_{j=1}^m \left[\frac{d\xi_1^{(j)}(x)}{dx} h_1^{(j)} + \frac{d\xi_2^{(j)}(x)}{dx} h_2^{(j)} + \dots + \frac{d\xi_{k_j}^{(j)}(x)}{dx} h_{k_j}^{(j)} \right] &= \\ = \sum_{j=1}^m \left[\xi_1^{(j)}(x) A h_1^{(j)} + \xi_2^{(j)}(x) A h_2^{(j)} + \dots + \xi_{k_j}^{(j)}(x) A h_{k_j}^{(j)} \right] &= \end{aligned}$$

$$= \sum_{j=1}^m \left[\xi_1^{(j)}(x) \lambda_j h_1^{(j)} + \xi_2^{(j)}(x) (\lambda_j h_2^{(j)} + h_1^{(j)}) + \dots + \xi_{k_j}^{(j)}(x) (\lambda_j h_{k_j}^{(j)} + h_{k_j-1}^{(j)}) \right] =$$

$$\sum_{j=1}^m \left[(\lambda_j \xi_1^{(j)}(x) + \xi_2^{(j)}(x)) h_1^{(j)} + (\lambda_j \xi_2^{(j)}(x) + \xi_3^{(j)}(x)) h_2^{(j)} + \dots \right.$$

$$\left. + (\lambda_j \xi_{k_j-1}^{(j)}(x) + \xi_{k_j}^{(j)}(x)) h_{k_j-1}^{(j)} + \xi_{k_j}^{(j)}(x) \lambda_j h_{k_j}^{(j)} \right]$$

munosabat kelib chiqadi. Har bir $y(x)$ vektorni Jordan bazisi bo'yicha (4.2.6) yoyilmasining yagonaligidan quyidagi

$$\frac{d\xi_1^{(j)}(x)}{dx} = \lambda_j \xi_1^{(j)}(x) + \xi_2^{(j)}(x),$$

$$\frac{d\xi_2^{(j)}(x)}{dx} = \lambda_j \xi_2^{(j)}(x) + \xi_3^{(j)}(x),$$

.....

$$\frac{d\xi_{k_j-1}^{(j)}(x)}{dx} = \lambda_j \xi_{k_j-1}^{(j)}(x) + \xi_{k_j}^{(j)}(x),$$

$$\frac{d\xi_{k_j}^{(j)}(x)}{dx} = \lambda_j \xi_{k_j}^{(j)}(x)$$

m ta differensial tenglamalar sistemasi kelib chiqadi. Bu sistemani quyidan yuqoriga qarab yechsak,

$$\xi_{k_j}^{(j)}(x) = C_{k_j}^{(j)} e^{\lambda_j x},$$

$$\xi_{k_j-1}^{(j)}(x) = \left[C_{k_j-1}^{(j)} + C_{k_j}^{(j)} \frac{x}{1!} \right] e^{\lambda_j x},$$

$$\xi_1^{(j)}(x) = \left[C_1^{(j)} + C_2^{(j)} \frac{x}{1!} + \dots + C_{k_j}^{(j)} \frac{x^{k_j-1}}{(k_j-1)!} \right] e^{\lambda_j x},$$

$$j = 1, 2, \dots, m. \quad j = \overline{1, m}$$

tengliklar hosil bo'ladi. Topilgan $\xi_{k_j}^{(j)}(x)$, $j = \overline{1, m}$ funksiyalarning bu ifodalarini (4.2.6) yoyilmaga qo'yib, $y(x)$ yechimning (4.2.5) ko'rinishda tasvirlanishiga ishonch hosil qilamiz.

Mustaqil yechish uchun mashqlar [21], §14, №726-752; [8], §22, №802-807.

3-§. O'zgaras koeffitsiyentli chiziqli bir jinsli bo'lmagan differensial tenglamalar sistemasi

Quyidagi

$$\frac{dy}{dx} = Ay + f(x), \quad x \in (a, b) \subseteq R \quad (4.3.1)$$

ko'rinishdagi differensial tenglamalar sistemasini qaraylik. Bu yerda $y(x) = (y_1(x), \dots, y_n(x))^T$ -noma'lum vektor-funksiya, $A = (a_{jk})$, $a_{jk} = \text{const}$ $j, k = \overline{1, n}$ -berilgan o'zgaras matritsa, $f(x) = (f_1(x), \dots, f_n(x))^T$ -berilgan uzluksiz vektor-funksiya.

Aytaylik, $y_0(x)$ vektor-funksiya (4.3.1) sistemaning biror yechimi bo'lsin. U holda (4.3.1) tenglamaga ushbu

$$y(x) = z(x) + y_0(x)$$

almashtirishni qo'llasak,

$$\frac{dz(x)}{dx} + \frac{dy_0(x)}{dx} = A(z(x) + y_0(x)) + f(x) = Az(x) + Ay_0(x) + f(x)$$

munosabat hosil bo'ladi. Shartga ko'ra $y_0(x)$ vektor-funksiya (4.3.1) tenglamani qanoatlantiradi, ya'ni

$$\frac{dy_0(x)}{dx} = Ay_0(x) + f(x).$$

Shuning uchun yuqoridagi tenglikdan ushbu

$$\frac{dz(x)}{dx} = Az(x) \quad (4.3.2)$$

bir jinsli differensial tenglama kelib chiqadi.

Agar (4.3.2) bir jinsli sistemaning $z_1(x), \dots, z_n(x)$ chiziqli erkli yechimlari ma'lum bo'lsa, u holda uning umumiy yechimi

$$z(x) = c_1 z_1(x) + c_2 z_2(x) + \dots + c_n z_n(x)$$

ko'rinishda bo'lishi oldindan ma'lum. Bu tenglikda c_j -ixtiyoriy haqiqiy o'zgaras sonlar. Demak, (4.3.1) sistemaning umumiy yechimi

$$y(x) = c_1 z_1(x) + c_2 z_2(x) + \dots + c_n z_n(x) + y_0(x) \quad (4.3.3)$$

ko'rinishda ifodalanar ekan.

4.3.1-lemma. Agar $y_1(x)$ va $y_2(x)$ vektor-funksiyalar mos ravishda quyidagi

$$\frac{dy_j(x)}{dx} = Ay_j(x) + f_j(x), \quad j = 1, 2$$

tenglamalarning yechimlari bo'lsa, u holda $y(x) = y_1(x) + y_2(x)$ vektor-funksiya ushbu

$$\frac{dy}{dx} = Ay + f_1(x) + f_2(x)$$

ko‘rinishdagi tenglamaning yechimidan iborat bo‘ladi.

Isbot.

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy_1}{dx} + \frac{dy_2}{dx} = Ay_1 + f_1(x) + Ay_2 + f_2(x) = \\ &= A(y_1 + y_2) + f_1(x) + f_2(x) = Ay + f_1(x) + f_2(x). \end{aligned}$$

Avvalo quyidagi xususiy hollarni ko‘rib chiqamiz.

1. Aytaylik, λ soni A matritsaning k karrali biror xos qiymati bo‘lib, unga h_1, h_2, \dots, h_k - Jordan zanjiri mos kelsin.

4.3.1-teorema. Faraz qilaylik, (4.3.1) tenglamada

$$f(x) = e^{\mu x} \left[P_m^{(1)}(x)h_1 + P_m^{(2)}(x)h_2 + \dots + P_m^{(k)}(x)h_k \right]$$

ko‘rinishdagi vektor-funksiya bo‘lsin. U holda (4.3.1) differensial tenglamalar sistemasining ushbu

$$y(x) = \begin{cases} e^{\mu x} Q_m(x), & \mu \neq \lambda, \\ x e^{\mu x} Q_{m+k-1}(x), & \mu = \lambda \end{cases}$$

ko‘rinishdagi yechimi mavjud va yagona. Bu yerda $P_m^{(1)}(x), P_m^{(2)}(x), \dots, P_m^{(k)}(x)$ darajasi m dan oshmaydigan berilgan ko‘phadlar, $Q_m(x)$ va $Q_{m+k-1}(x)$ - ko‘phadlarning darajalari mos ravishda m va $m+k-1$.

Isbot. (4.3.1) differensial tenglamalar sistemasining yechimini

$$y(x) = \sum_{j=1}^k \xi_j(x) h_j \tag{4.3.4}$$

ko‘rinishda izlaymiz. (4.3.4) vektor-funksiyani (4.3.1) tenglamaga qo‘yib, Jordan zanjirining ta’rifidan foydalansak,

$$\begin{aligned} \sum_{j=1}^k \xi'_j(x) h_j &= \sum_{j=1}^k \xi_j(x) A h_j + \\ &+ e^{\mu x} \sum_{j=1}^k P_m^{(j)}(x) h_j = \lambda \xi_1(x) h_1 + \\ &+ \sum_{j=2}^k \xi_j(x) [\lambda h_j + h_{j-1}] + e^{\mu x} \sum_{j=1}^k P_m^{(j)}(x) h_j \end{aligned}$$

munosabat hosil bo‘ladi. Bundan va h_1, h_2, \dots, h_k vektorlarning chiziqli erkliligidan quyidagi

$$\begin{aligned} \xi'_1(x) &= \lambda \xi_1(x) + \xi_2(x) + e^{\mu x} P_m^{(1)}(x), \\ &\dots\dots\dots \\ \xi'_{k-1}(x) &= \lambda \xi_{k-1}(x) + \xi_k(x) + e^{\mu x} P_m^{(k-1)}(x), \\ \xi'_k(x) &= \lambda \xi_k(x) + e^{\mu x} P_m^{(k)}(x) \end{aligned} \tag{4.3.5}$$

ko‘rinishdagi differensial tenglamalar kelib chiqadi. Bu tenglamalarni quyidan yuqoriga qarab yechamiz.

Agar $\lambda \neq \mu$ bo'lsa, u holda bu differensial tenglamalarning

$$\xi_k(x) = e^{\mu x} Q_m^{(k)}(x), \dots, \xi_1(x) = e^{\mu x} Q_m^{(1)}(x)$$

ko'rinishdagi xususiy yechimlari mavjud va yagona. Bu yerda $Q_m^{(1)}(x), \dots, Q_m^{(k)}(x)$ - ko'phadlarning kamida bittasining darajasi m ga teng. Topilgan $\xi_j(x)$ larning bu ifodalarini (4.3.4) yoyilmaga qo'yib, (4.3.1) differensial tenglamalar sistemasining yechimini

$$y(x) = e^{\mu x} \sum_{j=1}^k Q_m^{(j)}(x) h_j = e^{\mu x} Q_m(x)$$

ko'rinishda topamiz.

Agar $\lambda = \mu$ bo'lsa, u holda (4.3.5) tenglamalarning

$$\xi_k(x) = x e^{\mu x} Q_m^{(k)}(x), \xi_{k-1}(x) = x e^{\mu x} Q_{m+1}^{(k-1)}(x), \dots, \xi_1(x) = x e^{\mu x} Q_{m+k-1}^{(1)}(x)$$

ko'rinishdagi xususiy yechimlari mavjud va yagona. Bu yerda $Q_m^{(k)}(x), \dots, Q_{m+k-1}^{(1)}(x)$ mos ravishda $m, \dots, m+k-1$ darajali ko'phadlar. Topilgan $\xi_j(x)$ larning bu ifodalarini (3.4.4) yoyilmaga qo'yib,

$$y(x) = x e^{\mu x} \sum_{j=1}^k Q_{m+k-1}^{(j)}(x) h_j = e^{\mu x} x Q_{m+k-1}(x)$$

(4.3.1) differensial tenglamaning yechimini topamiz.

2. Endi umumiy holni qaraymiz.

4.3.2-teorema. Aytaylik, (4.3.1) sistemada

$$f(x) = e^{\mu x} P_m(x)$$

ko'rinishda bo'lsin. Bunda $P_m(x)$ m -darajali vektor ko'phad. U holda (4.3.1) differensial tenglamaning

$$y(x) = e^{\mu x} Q_{m+k}(x) \quad (4.3.6)$$

ko'rinishdagi yechimi mavjud. Bu yerda $Q_{m+k}(x) - (m+k)$ darajali vektor-ko'phad bo'lib, agar μ soni A matritsaning xos qiymati bo'lmasa, $k=0$, agar μ soni A matritsaning r karrali xos qiymatidan iborat bo'lsa, $k=r$ deb olinadi. Bundan tashqari $Q_{m+k}(x)$ ko'phadning koeffitsientlari n o'lchamli sonli vektorlardan iborat.

Isbot. Aytaylik, A matritsa k_j karrali λ_j xos qiymatga ega bo'lib, R^n fazoning Jordan bazisi m ta $h_1^{(j)}, h_2^{(j)}, \dots, h_k^{(j)}$, $j = \overline{1, m}$ - Jordan zanjiridan iborat bo'lsin. Ushbu $P_m(x)$ vektor-ko'phadni Jordan bazisi bo'yicha yoyamiz:

$$P_m(x) = \sum_{j=1}^m \left[P_1^{(j)}(x) h_1^{(j)} + P_2^{(j)}(x) h_2^{(j)} + \dots + P_{k_j}^{(j)}(x) h_{k_j}^{(j)} \right].$$

Bunda $P_1^{(j)}(x), \dots, P_{k_j}^{(j)}(x)$, $j = \overline{1, m}$ ko'phadlarning ichida kamida bittasi m -darajali. Agar

$$y^{(j)}(x) = \left(y_1^{(j)}(x), \dots, y_n^{(j)}(x) \right)^T$$

vektor-funksiya ushbu

$$\frac{dy^{(j)}}{dx} = Ay^{(j)} + e^{\mu x} \left[P_1^{(j)}(x)h_1^{(j)} + \dots + P_{k_j}^{(j)}(x)h_{k_j}^{(j)} \right] \quad (4.3.7)$$

ko‘rinishdagi differensial tenglamaning yechimi bo‘lsa, unda quyidagi

$$y(x) = \sum_{j=1}^m y^{(j)}(x)$$

funksiya (4.3.1) sistemaning xususiy yechimi bo‘ladi.

Faraz qilaylik, $\lambda_1, \lambda_2, \dots, \lambda_q$ ($1 \leq q \leq n$) lar A matritsaning xos qiymatlari bo‘lsin.

Agar $\mu \neq \lambda_j, \forall j = \overline{1, q}$ bo‘lsa, u holda (4.3.7) bir jinsli bo‘lmagan sistema 4.3.1-teoremaga asosan

$$y^{(j)}(x) = e^{\mu x} Q_{j,m}(x)$$

ko‘rinishdagi xususiy yechimga ega bo‘ladi. Bu yerda $Q_{j,m}(x)$ darajasi m ga teng vektor-ko‘phad.

Agar $\mu = \lambda_1$ bo‘lsa, u holda (4.3.7) tenglama 4.3.1-teoremaga asosan

$$y^{(j)}(x) = xe^{\mu x} Q_{j,m+k-1}(x)$$

ko‘rinishidagi xususiy yechimga ega bo‘ladi. Bunda $Q_{j,m+k-1}(x)$ -darajasi $(m+k-1)$ ga teng vektor-ko‘phad. Qaralayotgan holda (4.3.7) tenglamalarning qolganlari

$$y^{(j)}(x) = e^{\mu x} Q_{j,m}(x)$$

ko‘rinishdagi xususiy yechimga ega bo‘ladi. Bu yerda $Q_{j,m}(x)$ -darajasi m ga teng bo‘lgan vektor-ko‘phad. Bu mulohazalardan va superpozitsiya prinspidan (4.3.1) sistema (4.3.6) ko‘rinishdagi xususiy yechimga ega ekanligi kelib chiqadi.

4.3.1-izoh. Amaliyotda (4.3.6) ko‘rinishdagi yechimni aniqmas koeffitsiyentlar usulidan foydalanib ham topish mumkin.

4-§. O‘zgarmas koeffitsiyentli chiziqli bir jinsli bo‘lmagan differensial tenglamalar sistemasini aniqmas koeffitsiyentlar usulida yechish

Bir jinsli bo‘lmagan ushbu

$$\frac{dy}{dx} = Ay + e^{\mu x} P_m(x) \quad (4.4.1)$$

ko‘rinishdagi differensial tenglamalar sistemasini berilgan bo‘lsin. Bu yerda $P_m(x)$ -darajasi m ga teng bo‘lgan

$$P_m(x) = \sum_{j=1}^m P_j x^j \quad (4.4.2)$$

ko‘rinishdagi vektor-ko‘phad, $P_j, j = \overline{0, m}$, o‘zgarmas vektorlar, $\mu = const$.

Quyidagi ikki hol bo‘lishi mumkin.

1. Rezonansmas hol. Bunda μ soni $\Delta(\lambda) = \det(A - \lambda I) = 0$ xarakteristik tenglamaning ildizi bo‘lmaydi, ya’ni $\Delta(\mu) \neq 0$ bo‘ladi. U holda (4.4.1) differensial tenglamalar sistemasi

$$y(x) = e^{\mu x} Q_m(x) \quad (4.4.3)$$

ko‘rinishdagi xususiy yechimga ega bo‘ladi. Bunda $Q_m(x)$, m -darajali ko‘phad.

$Q_m(x)$ ko‘phadni quyidagi

$$Q_m(x) = \sum_{j=0}^m q_j x^j \quad (4.4.4)$$

ko‘rinishda izlaymiz. Bu yerda q_j -noma’lum o‘zgarmas vektorlar. (4.4.1) va (4.4.3) munosabatlardan

$$\mu e^{\mu x} Q_m(x) + e^{\mu x} \frac{dQ_m(x)}{dx} = A e^{\mu x} Q_m(x) + e^{\mu x} P_m(x)$$

tengliklarni hosil qilamiz. Bu tenglikning ikki tamonini $e^{\mu x} \neq 0$ ga bo‘lib, ushbu

$$\mu Q_m(x) - A Q_m(x) = P_m(x) - \frac{dQ_m(x)}{dx} \quad (4.4.5)$$

munosabatni topamiz. Avvalo (4.4.5) tenglikni quyidagi

$$\|\mu I - A\| Q_m(x) = P_m(x) - \frac{dQ_m(x)}{dx} \quad (4.4.6)$$

ko‘rinishda yozib olamiz. So‘ngra bu tenglikning ikki tomonidagi x^m, x^{m-1} darajalar oldidagi mos koeffitsiyentlarni tenglashtirsak,

$$\begin{aligned} \|\mu I - A\| q_m &= p_m, \\ \|\mu I - A\| q_{m-1} &= p_{m-1} - m q_m, \end{aligned} \quad (4.4.7)$$

tenglamalar sistemasi hosil bo‘ladi.

Shartga ko‘ra, $\Delta(\mu) \neq 0$ bo‘lgani uchun $\|\mu I - A\|^{-1}$ teskari matritsa mavjud. Shuning uchun (4.4.7) sistemaning birinchi tenglamasidan

$$q_m = \|\mu I - A\|^{-1} p_m \quad (4.4.8)$$

noma’lumni, ikkinchisidan esa

$$q_{m-1} = \|\mu I - A\|^{-1} p_{m-1} - m \left(\|\mu I - A\|^{-1} \right)^2 p_m \quad (4.4.9)$$

noma'lumni topamiz. Bu jarayonni davom ettirib, $q_i, i = \overline{0, m}$ noma'lum vektorlarning barchasini topish mumkin.

2. Rezonans hol. Bunda μ soni $\Delta(\lambda) = \det(A - \lambda I) = 0$ xarakteristik tenglamaning ildizi, ya'ni $\Delta(\mu) = 0$ bo'ladi. Bundan tashqari A matritsa faqat oddiy xos qiymatlarga ega bo'lsin. U holda (4.4.1) differensial tenglamalar sistemasining xususiy yechimi

$$y(x) = e^{\mu x} x \cdot Q_m(x) \quad (4.4.10)$$

ko'rinishida bo'ladi. Bu yerda $Q_m(x)$ – m darajali vektor-ko'phad. Avvalo ushbu

$$y(x) = Tz(x)$$

almashtirishdan foydalanib, (4.4.1) sistemani $n = 2$ holda

$$\begin{cases} \frac{dz_1}{dx} = \lambda_1 z_1 + e^{\mu x} f_1(x) \\ \frac{dz_2}{dx} = \lambda_2 z_2 + e^{\mu x} f_2(x) \end{cases} \quad (4.4.11)$$

ko'rinishga keltiramiz. Bu yerdagi $f_k(x)$ ko'phad $T^{-1}P_m(x)$ ning mos komponentasi. (4.4.11) tenglamalarning har biri uchun xususiy yechimni

$$z(x) = xQ_m(x)e^{\mu x} \equiv e^{\mu x} \cdot Q_{m+1}(x)$$

ko'rinishda izlaymiz. Bu yerda $Q_{m+1}(x)$ – $m+1$ darajali ko'phad.

5-§. Misollar yechish namunalari

4.5.1-misol. Ushbu

$$\begin{cases} \frac{dy_1}{dx} = y_2, \\ \frac{dy_2}{dx} = y_1 \end{cases}$$

bir jinsli differensial tenglamalar sistemasining yechimi topilsin.

Yechish. Avvalo berilgan differensial tenglamalar sistemasining matritsasini tuzib olamiz:

$$A = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}.$$

Bu matritsaning xos qiymatlarini va xos vektorlarini topamiz. Buning uchun ushbu

$$Ah = \lambda h, \quad h \neq 0, h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$$

tenglamani qaraymiz. Oxirgi tenglamani koordinatalarda yozamiz:

$$\begin{cases} h_2 = \lambda h_1, \\ h_1 = \lambda h_2, \end{cases} \quad \begin{cases} h_2 - \lambda h_1 = 0, \\ h_1 - \lambda h_2 = 0. \end{cases}$$

Ma'lumki, bir jinsli algebraik tenglamalar sistemasi nolmas yechimga ega bo'lishi uchun, uning asosiy determinantining nolga teng bo'lishi zarur va yetarli.

Shuning uchun

$$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0, \quad \lambda^2 - 1 = 0.$$

Oxirgi kvadrat tenglamani yechib, A matritsaning xos qiymatlarini topamiz: $\lambda_1 = 1$, $\lambda_2 = -1$. Endi $\lambda_1 = 1$ xos qiymatga mos keluvchi

$$h_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

xos vektorni topamiz. Ushbu

$$\begin{cases} -x_1 + x_2 = 0, \\ x_1 - x_2 = 0 \end{cases}$$

sistemasidan $x_1 = 1$, $x_2 = 1$, ya'ni $h_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ xos vektorni topamiz.

$\lambda_2 = -1$ xos qiymatga mos keluvchi h_2 xos vektorni topish uchun ushbu

$$\begin{cases} x_1 + x_2 = 0, \\ x_1 + x_2 = 0 \end{cases}$$

sistemani hosil qilamiz. Bu sistemadan $x_1 = 1$, $x_2 = -1$, ya'ni $h_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ xos vektorni topamiz. Endi berilgan differensial tenglamalar sistemasining ixtiyoriy yechimini quyidagicha yozish mumkin:

$$\begin{aligned} y(x) &= C_1 e^{\lambda_1 x} h_1 + C_2 e^{\lambda_2 x} h_2 = \\ &= C_1 e^x h_1 + C_2 e^{-x} h_2 = C_1 e^x \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-x} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \\ &= \begin{pmatrix} C_1 e^x + C_2 e^{-x} \\ C_1 e^x - C_2 e^{-x} \end{pmatrix}, \end{aligned}$$

ya'ni

$$\begin{aligned} y_1(x) &= C_1 e^x + C_2 e^{-x}, \\ y_2(x) &= C_1 e^x - C_2 e^{-x}. \end{aligned}$$

Bunda C_1, C_2 – ixtiyoriy haqiqiy o'zgarmas sonlar.

4.5.2-misol. Ushbu

$$\begin{cases} \frac{dy_1}{dx} = 3y_1 - 2y_2, \\ \frac{dy_2}{dx} = y_1 + y_2 \end{cases}$$

differensial tenglamalar sistemasining yechimini toping.

Yechish. Berilgan sistemaning koeffitsiyentlaridan

$$A = \begin{vmatrix} 3 & -2 \\ 1 & 1 \end{vmatrix}$$

matritsa tuzib, uning xos qiymatlarini hamda xos vektorlarini topamiz:

$$Ah = \lambda h, \quad h = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

$$\begin{cases} 3x_1 - 2x_2 = \lambda x_1, \\ x_1 + x_2 = \lambda x_2, \end{cases}$$

$$\begin{cases} (3 - \lambda)x_1 - 2x_2 = 0, \\ x_1 + (1 - \lambda)x_2 = 0, \end{cases} \quad \begin{vmatrix} 3 - \lambda & -2 \\ 1 & 1 - \lambda \end{vmatrix} = 0.$$

Ushbu $\lambda^2 - 4\lambda + 5 = 0$ kvadrat tenglamani yechib, $\lambda_{1,2} = 2 \pm i$ xos qiymatlarni

topib olamiz. Endi $\lambda_1 = 2 + i$ xos qiymatga mos keluvchi $h = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ xos

vektorni topish bilan shug'ullanamiz:

$$\begin{cases} (1 - i)x_1 - 2x_2 = 0, \\ x_1 - (1 + i)x_2 = 0. \end{cases}$$

Bunda $x_2 = 1$ deb, $x_1 = 1 + i$ ni topamiz. Bu holda xos vektor

$$h_1 = \begin{pmatrix} 1 + i \\ 1 \end{pmatrix}$$

ko'rinishida bo'ladi. Bu xos qiymatga berilgan sistemaning

$$y(x) = e^{(2+i)x} h_1 = e^{(2+i)x} \begin{pmatrix} 1 + i \\ 1 \end{pmatrix}$$

ko'rinishidagi xususiy yechimi mos keladi. Bu xususiy yechimning haqiqiy va mavhum qismlari ham berilgan differensial tenglamalar sistemasining xususiy yechimlari bo'ladi:

$$\operatorname{Re}\{y(x)\} = \operatorname{Re}\{e^{(2+i)x} h_1\} = \operatorname{Re}\left\{e^{(2+i)x} \begin{pmatrix} 1 + i \\ 1 \end{pmatrix}\right\} = e^{2x} \begin{pmatrix} \cos x - \sin x \\ \cos x \end{pmatrix},$$

$$\operatorname{Im}\{y(x)\} = \operatorname{Im}\{e^{(2+i)x} h_1\} = \operatorname{Im}\left\{e^{(2+i)x} \begin{pmatrix} 1 + i \\ 1 \end{pmatrix}\right\} = e^{2x} \begin{pmatrix} \cos x + \sin x \\ \sin x \end{pmatrix},$$

$$y(x) = C_1 \operatorname{Re}\{e^{(2+i)x} h_1\} + C_2 \operatorname{Im}\{e^{(2+i)x} h_1\} = \\ = C_1 \begin{pmatrix} \cos x - \sin x \\ \cos x \end{pmatrix} e^{2x} + C_2 \begin{pmatrix} \cos x + \sin x \\ \sin x \end{pmatrix} e^{2x}.$$

Bu yerda C_1, C_2 -ixtiyoriy haqiqiy o'zgarmas sonlar.

4.5.3-misol. Ushbu

$$\begin{cases} \frac{dy_1}{dx} = 2y_1 + y_2, \\ \frac{dy_2}{dx} = -y_1 + 4y_2 \end{cases}$$

bir jinsli differensial tenglamalar sistemasining yechimini toping.

Yechish. Berilgan differensial tenglamalar sistemasining koeffitsiyentlaridan

$$A = \begin{vmatrix} 2 & 1 \\ -1 & 4 \end{vmatrix}$$

matritsani tuzib olamiz va uning xos qiymatlarini hamda xos vektorlarini hisoblaymiz. Buning uchun

$$Ah = \lambda h, \quad 0 \neq h = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

tenglamani qaraymiz va uni koordinatalarda yozib, quyidagi

$$\begin{cases} 2x_1 + x_2 = \lambda x_1, \\ -x_1 + 4x_2 = \lambda x_2 \end{cases}$$

sistemani hosil qilamiz. Bundan

$$\begin{cases} (2 - \lambda)x_1 + x_2 = 0, \\ -x_1 + (4 - \lambda)x_2 = 0 \end{cases}$$

bir jinsli tenglamalar sistemasi kelib chiqadi. Ma'lumki, bir jinsli sistema nolmas yechimga ega bo'lishi uchun, uning asosiy determinantining nolga teng bo'lishi zarur va yetarli. Shuning uchun

$$\begin{vmatrix} 2 - \lambda & 1 \\ -1 & 4 - \lambda \end{vmatrix} = 0, \quad \lambda^2 - 6\lambda + 9 = 0$$

xarakteristik tenglamani yechib, $\lambda_1 = \lambda_2 = 3$ karrali xos qiymatni topamiz.

Endi $\lambda_1 = 3$ xos qiymatga mos keluvchi $h_1 = (x_1, x_2)^T$ -xos vektorni aniqlaymiz. Bunda $r(A) = 1$ bo'lgani uchun,

$$(2 - 3)x_1 + x_2 = 0, \quad -x_1 + x_2 = 0$$

tenglamadan $x_1 = 1$ deb, $x_2 = 1$ ni, ya'ni

$$h_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

xos vektorni topamiz.

Nihoyat, $\lambda = 3$ karrali xos qiymatga mos keluvchi yopishgan(ergashgan) vektorni topamiz. Buning uchun ushbu

$$Ah_2 = 3h_2 + h_1$$

tenglamani qaraymiz. Bu bir jinsli tenglamani koordinatalarda yozamiz:

$$\begin{cases} 2a + b = 3a + 1, \\ -a + 4b = 3b + 1, \end{cases} \quad \begin{cases} b - a = 1, \\ b - a = 1. \end{cases}$$

Bunda $a = 0$ deb, $b = 1$ ga, ya'ni

$$h_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

yopishgan vektorga ega bo'lamiz.

Yuqoridagi mulohazalar asosida berilgan differensial tenglamalar sistemasining xususiy yechimlari

$$y^{(1)}(x) = e^{3x}h_1 = e^{3x} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} e^{3x} \\ e^{3x} \end{pmatrix},$$

$$y^{(2)}(x) = e^{3x}(h_2 + xh_1) = e^{3x} \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} + x \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right] = e^{3x} \begin{pmatrix} x \\ x+1 \end{pmatrix}$$

ko'rinishida bo'lishiga ishonch hosil qilamiz.

Demak, berilgan sistemaning ixtiyoriy yechimi

$$\begin{aligned} y(x) &= C_1 y^{(1)}(x) + C_2 y^{(2)}(x) = \begin{pmatrix} C_1 e^{3x} + C_2 x e^{3x} \\ C_1 e^{3x} + C_2 (x+1) e^{3x} \end{pmatrix} = \\ &= e^{3x} \begin{pmatrix} C_1 + C_2 x \\ C_1 + C_2 (x+1) \end{pmatrix} \end{aligned}$$

ko'rinishida bo'lar ekan. Bunda C_1, C_2 -ixtiyoriy haqiqiy o'zgarmas sonlar.

4.5.4-misol. Quyidagi

$$\begin{cases} \frac{dy_1}{dx} = y_2 + y_3, \\ \frac{dy_2}{dx} = y_1 + y_3, \\ \frac{dy_3}{dx} = y_1 + y_2 \end{cases}$$

differensial tenglamalar sistemasining yechimi topilsin.

Yechish. Berilgan sistemaning koeffitsiyentlaridan

$$A = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

matritsani tuzib, uning xos qiymatlarini va xos vektorlarini topamiz. Buning uchun ushbu

$$Ah = \lambda h, \quad 0 \neq h = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

tenglamani qaraymiz. Bu tenglamani koordinatalarda yozib

$$\begin{cases} x_2 + x_3 = \lambda x_1, \\ x_1 + x_3 = \lambda x_2, \\ x_1 + x_2 = \lambda x_3 \end{cases} \quad \begin{cases} -\lambda x_1 + x_2 + x_3 = 0, \\ x_1 - \lambda x_2 + x_3 = 0, \\ x_1 + x_2 - \lambda x_3 = 0 \end{cases}$$

bir jinsli algebraik tenglamalar sistemasini hosil qilamiz. Ma'lumki bir jinsli sistema noldan farqli yechimga ega bo'lishi uchun, uning asosiy deterinanti nolga teng bo'lishi zarur va yetarli. Shuning uchun

$$\begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0, \quad \lambda^3 - 3\lambda - 2 = 0$$

xarakteristik tenglamani yechib, $\lambda_1 = 2$, $\lambda_2 = \lambda_3 = -1$ xos qiymatlarni topib

olamiz. So'ngra $\lambda_1 = 2$, xos qiymatga mos keluvchi $h_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ xos vektorni

topamiz. Buning uchun ushbu

$$\begin{cases} -2x_1 + x_2 + x_3 = 0, \\ x_1 - 2x_2 + x_3 = 0, \\ x_1 + x_2 - 2x_3 = 0 \end{cases}$$

sistemaning yechimini topamiz. Bunda $r(A)=2$ bo'lgani uchun quyidagi

$$\begin{cases} x_2 + x_3 = 2x_1, \\ -2x_2 + x_3 = -x_1 \end{cases}$$

sistemada $x_1 = 1$ deb, x_2 va x_3 larni quyidagi

$$\begin{cases} x_2 + x_3 = 2, \\ -2x_2 + x_3 = -1 \end{cases}$$

sistemadan topamiz: $x_2 = 1, x_3 = 1$. Demak, $\lambda_1 = 2$ xos qiymatga $h_1 = (1,1,1)^T$ xos vektor mos kelar ekan. Berilgan differensial tenglamalar sistemasining bu xos qiymatga mos keluvchi xususiy yechimi

$$y_1(x) = e^{2x} h_1 = e^{2x} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

ko‘rinishida bo‘ladi.

Endi $\lambda_2 = -1$ xos qiymatga mos keluvchi

$$h_2 = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

xos vektorni topamiz:

$$\begin{cases} a_1 + a_2 + a_3 = 0, \\ a_1 + a_2 + a_3 = 0, \\ a_1 + a_2 + a_3 = 0. \end{cases}$$

Bunda $a_1 = 1$, $a_2 = 0$ deb, $a_3 = -1$ ni topamiz. Demak, $\lambda_2 = -1$ xos qiymatga

$$h_2 = (1, 0, -1)^T$$

xos vektor mos kelar ekan. Endi ushbu $h_3 = (0, 1, -1)^T$ vektorni tekshiraylik:

$$Ah_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = -1 \cdot \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = -1 \cdot h_3.$$

Demak, h_3 vektor ham $\lambda_2 = -1$ karrali xos qiymatga mos keluvchi xos vektor bo‘lar ekan. Endi ushbu

$$ah_2 + bh_3 = 0 \quad \forall a, b = \text{const}$$

munosabatni qaraylik. Bundan

$$\begin{pmatrix} a \\ 0 \\ -a \end{pmatrix} + \begin{pmatrix} 0 \\ b \\ -b \end{pmatrix} = 0, \quad \begin{pmatrix} a \\ b \\ -a-b \end{pmatrix} = 0 \Rightarrow a = 0, b = 0$$

kelib chiqadi. Demak, h_2, h_3 vektorlar chiziqli erkli ekan. Bu holda yopishgan vektor h_3 xos vektor bilan ustma-ust tushadi. Shuning uchun

$$y_2(x) = e^{-x} h_2 = e^{-x} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad y_3(x) = e^{-x} h_3 = e^{-x} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

vektor-funksiyalar berilgan differensial tenglamalar sistemasining xususiy yechimlari bo‘ladi. Ushbu

$$y(x) = C_1 y_1(x) + C_2 y_2(x) + C_3 y_3(x) = C_1 e^{2x} h_1 + C_2 e^{-x} h_2 + C_3 e^{-x} h_3$$

vektor funksiya berilgan sistemaning umumiy yechimini beradi. Bunda C_1, C_2, C_3 – ixtiyoriy haqiqiy o‘zgarmaslar.

4.5.5-misol. Quyidagi

$$\begin{cases} y_1' = -2y_2 + 2y_3, \\ y_2' = y_1 - y_2 + y_3, \\ y_3' = y_2 - y_3 \end{cases}$$

differensial tenglamalar sistemasining kompleks qiymatli yechimini toping.

Yechish. Berilgan differensial tenglamalar sistemasining koeffitsiyentlaridan

$$A = \begin{vmatrix} 0 & -2 & 2 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{vmatrix}$$

matritsani tuzib olamiz va uning xos qiymatlarini hamda xos vektorlarini topamiz.

Buning uchun

$$Ah = \lambda h, \quad 0 \neq h = (x_1, x_2, x_3)^T$$

tenglamani qaraymiz. Bu tenglamani koordinatalarda yozib,

$$\begin{cases} -2x_2 + 2x_3 = \lambda x_1, \\ x_1 - x_2 + x_3 = \lambda x_2, \\ x_2 - x_3 = \lambda x_3, \end{cases} \quad \begin{cases} -\lambda x_1 - 2x_2 + 2x_3 = 0, \\ x_1 + (-1 - \lambda)x_2 + x_3 = 0, \\ x_2 + (-1 - \lambda)x_3 = 0. \end{cases}$$

bir jinsli algebraik tenglamalar sistemasini hosil qilamiz. Bu sistemaning nolmas yechimini topish maqsadida

$$\begin{vmatrix} -\lambda & -2 & 2 \\ 1 & -1 - \lambda & 1 \\ 0 & 1 & -1 - \lambda \end{vmatrix} = 0, \quad -\lambda(\lambda^2 + 2\lambda + 2) = 0$$

xarakteristik tenglamani olamiz va uni yechib, $\lambda_1 = 0, \lambda_{2,3} = -1 \pm i$ xos qiymatlarni aniqlaymiz. Xuddi oldingi misollardagi kabi, quyidagi

$$h_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad h_2 = \begin{pmatrix} 2 \\ -i \\ -1 \end{pmatrix}, \quad h_3 = \begin{pmatrix} 2 \\ i \\ -1 \end{pmatrix}$$

xos vektorlarni topamiz. Bunda $\lambda_1 = 0, \lambda_2 = -1 + i, \lambda_3 = -1 - i$ xos qiymatlar har xil bo'lgani uchun h_1, h_2, h_3 vektorlar R^3 fazoda bazis tashkil qiladi. Shuning uchun berilgan differensial tenglamalar sistemasining yechimi quyidagi

$$y(x) = (y_1(x), y_2(x), y_3(x))^T = C_1 h_1 + C_2 e^{(1+i)x} h_2 + C_3 e^{(-1-i)x} h_3 =$$

$$= C_1 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 2 \\ -i \\ -1 \end{pmatrix} e^{(-1+i)x} + C_3 \begin{pmatrix} 2 \\ i \\ -1 \end{pmatrix} e^{(-1-i)x}$$

vektor funksiyadan iborat bo‘ladi.

4.5.6-misol. Ushbu

$$\begin{cases} \frac{dy_1}{dx} = 4y_1 - y_2 - y_3, \\ \frac{dy_2}{dx} = y_1 + 2y_2 - y_3, \\ \frac{dy_3}{dx} = y_1 - y_2 + 2y_3 \end{cases}$$

differensial tenglamalar sistemasining yechimi topilsin.

Yechish. Quyidagi

$$A = \begin{vmatrix} 4 & -1 & -1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix}$$

matritsani tuzib olamiz va uning xos qiymatlarini hamda xos vektorlarini topamiz.

Avvalo $\Delta(\lambda) = \det(A - \lambda I) = 0$, ya'ni

$$\begin{vmatrix} 4 - \lambda & -1 & -1 \\ 1 & 2 - \lambda & -1 \\ 1 & -1 & 2 - \lambda \end{vmatrix} = 0, \quad (3 - \lambda)(\lambda^2 - 5\lambda + 6) = 0$$

xarakteristik tenglamani yechib, $\lambda_1 = 2, \lambda_{2,3} = 3$ xos qiymatlarni topamiz.

So'ngra $\lambda_1 = 2$ xos qiymatga mos keluvchi $h_1 = (1, 1, 1)^T$ xos vektorni,

keyinchalik $\lambda_2 = \lambda_3 = 3$ karrali xos qiymatlarga mos keluvchi $h_2 = (1, 1, 0)^T$,

$h_3 = (1, 0, 1)^T$ chiziqli erkli xos vektorlarni topamiz. Bu h_1, h_2, h_3 vektorlar R^3 fazoning bazisini tashkil qiladi. Shuning uchun berilgan sistemaning yechimi

$$y(x) = C_1 e^{2x} h_1 + C_2 e^{3x} h_2 + C_3 e^{3x} h_3$$

ko'rinishida bo'ladi. Bunda C_1, C_2, C_3 -ixtiyoriy haqiqiy o'zgarmas sonlar.

4.5.7-misol. Ushbu

$$\begin{cases} y_1'(x) = y_1 - 2y_2, \\ y_2'(x) = -y_1 - y_2 - 2y_3, \\ y_3'(x) = y_2 + y_3 \end{cases}$$

differensial tenglamalar sistemasiga qo'yilgan

$$y_1(0) = 1, \quad y_2(0) = -1, \quad y_3(0) = 1$$

Koshi masalasining yechimini toping.

Yechish. Quyidagi

$$A = \begin{vmatrix} 1 & -2 & 0 \\ -1 & -1 & -2 \\ 0 & 1 & 1 \end{vmatrix}$$

matritsani tuzib olamiz va uning xos qiymatlarini hamda xos vektorlarini aniqlaymiz. Xos qiymatlarni ushbu $\Delta(\lambda) = \det(A - \lambda I) = 0$, ya'ni

$$\begin{vmatrix} 1 - \lambda & -2 & 0 \\ -1 & -1 - \lambda & -2 \\ 0 & 1 & 1 - \lambda \end{vmatrix} = 0$$

xarakteristik tenglamadan aniqlaymiz: $\lambda_1 = -1$, $\lambda_2 = \lambda_3 = 1$. Endi $\lambda_1 = -1$ xos qiymatga mos keluvchi $h_1 = (2, 2, -1)^T$ xos vektorni topamiz. So'ngra $\lambda_2 = 1$ xos qiymatga mos keluvchi $h_2 = (2, 0, -1)^T$ xos vektorni aniqlaymiz. Bu h_2 vektorga yopishgan h_3 vektorni aniqlash uchun

$$Ah_3 = \lambda_2 h_3 + h_2$$

tenglamaga murojat qilamiz. Bundan $h_3 = (0, -1, 1)^T$ yopishgan vektorni topamiz. Topilgan h_1, h_2, h_3 vektorlar R^3 fazoda Jordan bazisini tashkil qiladi. Shuning uchun berilgan differensial tenglamalar sistemasining yechimi

$$y(x) = C_1 e^{-x} h_1 + C_2 e^x h_2 + C_3 e^x [x h_2 + h_3]$$

ko'rinishida bo'ladi. Bunda C_j , $j = \overline{1, 3}$ - ixtiyoriy haqiqiy o'zgarmas sonlar.

Endi berilgan boshlang'ich shartlardan foydalanib, C_1, C_2, C_3 o'zgarmaslarning qiymatlarini topamiz:

$$\begin{cases} 2C_1 + 2C_2 = 0, \\ 2C_1 - C_3 = -1, \\ -C_1 - C_2 + C_3 = 1, \end{cases} \quad \begin{cases} C_1 + C_2 = 0, \\ 2C_1 - C_3 = 0, \\ -C_1 - C_2 + C_3 = 0, \end{cases} \quad \begin{cases} C_2 = 0, \\ C_1 = 0, \\ C_3 = 1. \end{cases}$$

Demak, Koshi masalasining yechimi

$$y(x) = e^x [x h_2 + h_3] = e^x \left[\begin{pmatrix} 2x \\ 0 \\ -x \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right] = e^x \begin{pmatrix} 2x \\ -1 \\ -x + 1 \end{pmatrix}$$

ko'rinishidagi vektor-funksiyadan iborat bo'lar ekan.

4.5.8-misol. Ushbu

$$\begin{cases} y'_1 = 2y_1 - 5y_2 - 8y_3, \\ y'_2 = 7y_1 - 11y_2 - 17y_3, \\ y'_3 = -3y_1 + 4y_2 + 6y_3 \end{cases}$$

differensial tenglamalar sistemasining yechimi topilsin.

Yechish. Xuddi oldingi misollardagi kabi quyidagi

$$A = \begin{vmatrix} 2 & -5 & -8 \\ 7 & -11 & -17 \\ -3 & 4 & 6 \end{vmatrix}$$

matritsani tuzib, uning xos qiymatlarini va xos vektorlarini topamiz. Buning uchun ushbu

$$Ah = \lambda h, \quad 0 \neq h = (x_1, x_2, x_3)^T$$

tenglamani qaraymiz. Bundan

$$\begin{cases} 2x_1 - 5x_2 - 8x_3 = \lambda x_1, \\ 7x_1 - 11x_2 - 17x_3 = \lambda x_2, \\ -3x_1 + 4x_2 + 6x_3 = \lambda x_3, \end{cases}$$

ya'ni

$$\begin{cases} (2 - \lambda)x_1 - 5x_2 - 8x_3 = 0, \\ 7x_1 + (-11 - \lambda)x_2 - 17x_3 = 0, \\ -3x_1 + 4x_2 + (6 - \lambda)x_3 = 0 \end{cases}$$

bir jinsli tenglamalar sistemasi kelib chiqadi. A matritsaning xos qiymatlari ushbu

$$\Delta(\lambda) = \det(A - \lambda I) = 0,$$

$$\Delta(\lambda) = \begin{vmatrix} 1 - \lambda & -2 & 0 \\ -1 & -1 - \lambda & -2 \\ 1 & 1 & 1 - \lambda \end{vmatrix} = 0$$

xarakteristik tenglamaning ildizlaridan iborat:

$$\begin{aligned} \Delta(\lambda) &= (2 - \lambda)(-11 - \lambda)(6 - \lambda) - 8 \cdot 7 \cdot 4 - 5 \cdot 3 \cdot 17 - (-3)(-8)(-11 - \lambda) - \\ &\quad - 4(-17)(2 - \lambda) - 7(-5)(6 - \lambda) = \\ &= (\lambda - 2)(\lambda + 11)(6 - \lambda) - 224 - 255 + 24 \cdot (11 + \lambda) + 68(2 - \lambda) + 35(6 - \lambda) = \\ &= (\lambda^2 + 9\lambda - 22)(6 - \lambda) + 131 - 79\lambda = \\ &= 6\lambda^2 - \lambda^3 + 54\lambda - 9\lambda^2 - 132 + 22\lambda + 131 - 79\lambda = -\lambda^3 - 3\lambda^2 - 3\lambda - 1, \\ \Delta(\lambda) &= 0, \quad -\lambda^3 - 3\lambda^2 - 3\lambda - 1 = 0, \\ (\lambda + 1)^3 &= 0, \quad \lambda_1 = \lambda_2 = \lambda_3 = -1. \end{aligned}$$

Avvalo $\lambda_1 = -1$ xos qiymatga mos keluvchi xos vektorni topamiz. Buning uchun ushbu

$$\begin{cases} 3x_1 - 5x_2 - 8x_3 = 0, \\ 7x_1 - 10x_2 - 17x_3 = 0, \\ -3x_1 + 4x_2 + 7x_3 = 0 \end{cases}$$

sistemani tuzib olamiz. Ko‘rinib turibdiki ushbu

$$h = C \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad \forall C = \text{const}$$

vektorning koordinatalari bu sistemani qanoatlantiradi. Shuning uchun $C = 1$ deb $h_1 = (1, -1, 1)^T$ xos vektorni tanlab olamiz. So‘ngra h_1 xos vektorga yopishgan h_2 -vektorni ushbu

$$Ah_2 = (-1)h_2 + h_1$$

tenglamadan topib olamiz. Bu bir jinslimas tenglama quyidagi

$$\begin{cases} 3x_1 - 5x_2 - 8x_3 = 1, \\ 7x_1 - 10x_2 - 17x_3 = -1, \\ -3x_1 + 4x_2 + 7x_3 = 1 \end{cases}$$

ko‘rinishni oladi. Bundan $x_1 = -3, x_2 = -2, x_3 = 0$, ya‘ni $h_2 = (-3, -2, 0)^T$ yopishgan vektorni topamiz. Endi h_2 vektorga yopishgan h_3 vektorni ushbu

$$Ah_3 = (-1)h_3 + h_2$$

tenglamadan topib olamiz. Jumladan h_3 sifatida ushbu

$$h_3 = (5, 2, 1)^T$$

vektorni olish mumkin. Shunday qilib, R^3 fazoning Jordan bazisi h_1, h_2, h_3 vektorlardan iborat bo‘lar ekan. Bunga asosan berilgan differensial tenglamalar sistemasining yechimini topishimiz mumkin:

$$y(x) = C_1 e^{-x} h_1 + C_2 e^{-x} [x h_1 + h_2] + C_3 e^{-x} \left[\frac{x^2}{2} h_1 + x h_2 + h_3 \right].$$

Bu yerda C_1, C_2, C_3 –ixtiyoriy haqiqiy o‘zgarmas sonlar.

4.5.9-misol. Ushbu

$$\begin{cases} \frac{dy_1}{dx} = 2y_1 + y_2 + 2e^x, \\ \frac{dy_2}{dx} = y_1 + 2y_2 - 3e^x \end{cases}$$

ko‘rinishdagi bir jinsli bo‘lmagan differensial tenglamalar sistemasining yechimini toping.

Yechish. Berilgan tenglamalar sistemasiga mos keluvchi bir jinsli differensial tenglamalar sistemasini tuzib olamiz:

$$\text{a) } \begin{cases} \frac{dy_1}{dx} = 2y_1 + y_2, \\ \frac{dy_2}{dx} = y_1 + 2y_2. \end{cases}$$

Bu sistemaning koeffitsiyentlaridan

$$A = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}$$

matritsani tuzib, uning xos qiymatlarini va xos vektorlarini topamiz:

$$\Delta(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 2 - \lambda & 1 \\ 1 & 2 - \lambda \end{vmatrix} = 0, \quad (2 - \lambda)^2 - 1 = 0,$$

$$4 - 4\lambda + \lambda^2 - 1 = 0, \quad \lambda^2 - 4\lambda + 3 = 0, \quad \lambda_{1,2} = 2 \pm \sqrt{4 - 3} = 2 \pm 1, \quad \lambda_1 = 1, \lambda_2 = 3.$$

Endi $\lambda_1 = 1$ xos qiymatga mos keluvchi xos vektorni aniqlaymiz.

Buning uchun ushbu

$$Ah_1 = \lambda_1 h_1, \quad 0 \neq h_1 = (x_1, x_2)^T$$

tenglamani qaraymiz. Bundan

$$\begin{cases} x_1 + x_2 = 0, \\ x_1 + x_2 = 0 \end{cases}$$

sistemani keltirib chiqaramiz. Bu yerda $x_1 = 1$ deb, $x_2 = -1$ ni topamiz.

Natijada ushbu $h_1 = (1, -1)^T$ vektor $\lambda_1 = 1$ xos qiymatga mos keluvchi xos vektor bo'ladi. Xuddi shuningdek, $\lambda_2 = 3$ xos qiymatiga mos keluvchi

$$h_2 = (x_1, x_2)^T$$

xos vektorni topamiz. Ushbu

$$-x_1 + x_2 = 0$$

tenglamadan $x_1 = 1, x_2 = 1$ larni topib, $h_2 = (1, 1)^T$ xos vektorga ega bo'lamiz.

Bu mulohazalar asosida bir jinsli a) sistemaning umumiy yechimini topish mumkin:

$$y_u(x) = C_1 e^x h_1 + C_2 e^{3x} h_2.$$

Berilgan bir jinsli bo'lmagan sistemada $\mu = 1 = \lambda_1$, rezonans hol bo'lgani uchun uning xususiy yechimini

$$\bar{y}(x) = (a + bx)e^x$$

ko'rinishda izlaymiz. Bu yerda

$$a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

ko'rinishidagi vektorlar. Shuning uchun

$$\bar{y}(x) = (\bar{y}_1, \bar{y}_2)^T = \begin{pmatrix} a_1 + b_1x \\ a_2 + b_2x \end{pmatrix} e^x,$$

$$\bar{y}_1(x) = (a_1 + b_1x)e^x, \quad \bar{y}_2(x) = (a_2 + b_2x)e^x,$$

$$\bar{y}'_1(x) = e^x(a_1 + b_1x) + e^x b_1, \quad \bar{y}'_2(x) = e^x(a_2 + b_2x) + e^x b_2,$$

$$\begin{cases} e^x(a_1 + b_1x) + e^x b_1 = 2e^x(a_1 + b_1x) + (a_2 + b_2x)e^x + 2e^x \\ e^x(a_2 + b_2x) + e^x b_2 = (a_1 + b_1x)e^x + 2(a_2 + b_2x)e^x - 3e^x. \end{cases}$$

Bu munosabatning ikki tomonini $e^x \neq 0$ ga bo'lib,

$$\begin{cases} a_1 + b_1x + b_1 = 2a_1 + 2b_1x + a_2 + b_2x + 2, \\ a_2 + b_2x + b_2 = a_1 + b_1x + 2a_2 + 2b_2x - 3, \\ a_1 + b_1x + b_1 = 2a_1 + a_2 - 2 + (2b_1 + b_2)x, \\ a_2 + b_2x + b_2 = a_1 + 2a_2 - 3 + (b_1 + 2b_2)x \end{cases}$$

sistemani hosil qilamiz. Bunda ko'phadlarning mos koeffitsiyentlarini o'zaro tenglashtirib,

$$\begin{cases} a_1 + b_1 = 2a_1 + a_2 - 2, \\ b_1 = 2b_1 + b_2, \\ a_2 + b_2 = a_1 + 2a_2 - 3, \\ b_2 = b_1 + 2b_2 \end{cases}$$

tenglamalar sistemasini hosil qilamiz. Bundan

$$a = \left(0, -\frac{1}{2}\right)^T, \quad b = \left(-\frac{5}{2}, \frac{5}{2}\right)^T$$

aniqmas koeffitsiyentlarni topamiz. Demak, berilgan bir jinsli mas differensial tenglamaning xususiy yechimi

$$\bar{y}_1(x) = [a + bx]e^x = \left[\begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix} + \begin{pmatrix} -\frac{5}{2} \\ \frac{5}{2} \end{pmatrix} \cdot x \right] e^x$$

ko'rinishda bo'lar ekan. Berilgan sistemaning umumiy yechimi

$$y(x) = y_u(x) + \bar{y}(x) = C_1 e^x h_1 + C_2 e^{3x} h_2 + [a + bx]e^x =$$

$$C_1 e^x \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 e^{3x} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \left[\begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix} + \begin{pmatrix} -\frac{5}{2} \\ \frac{5}{2} \end{pmatrix} \cdot x \right] e^x$$

ko'rinishda bo'ladi.

4.5.10-misol. Ushbu

$$\begin{cases} y_1' = 4y_1 + 3y_2 - 3y_3, \\ y_2' = -3y_1 - 2y_2 + 3y_3, \\ y_3' = 3y_1 + 3y_2 - 3y_3 + 2e^{-x} \end{cases}$$

bir jinsli bo'lmagan differensial tenglamalar sistemasining yechimini toping.

Yechish. Berilgan differensial tenglamalar sistemasiga mos keluvchi bir jinsli sistemani olamiz:

$$\begin{cases} y_1' = 4y_1 + 3y_2 - 3y_3 \\ y_2' = -3y_1 - 2y_2 + 3y_3 \\ y_3' = 3y_1 + 3y_2 - 3y_3. \end{cases}$$

Bu sistemaning koeffitsiyentlaridan

$$A = \begin{vmatrix} 4 & 3 & -3 \\ -3 & -2 & 3 \\ 3 & 3 & -3 \end{vmatrix}$$

matritsa tuzib, uning xos qiymatlarini va xos vektorlarini topamiz. Buning uchun

$$Ah = \lambda h, \quad 0 \neq h = (x_1, x_2, x_3)^T$$

tenglamani qaraymiz. Bu tenglamani koordinatalarda yozib, ushbu

$$\begin{cases} 4x_1 + 3x_2 - 3x_3 = \lambda x_1, \\ -3x_1 - 2x_2 + 3x_3 = \lambda x_2, \\ 3x_1 + 3x_2 - 3x_3 = \lambda x_3, \end{cases}$$

ya'ni

$$\begin{cases} (4 - \lambda)x_1 + 3x_2 - 3x_3 = 0, \\ -3x_1 + (-2 - \lambda)x_2 + 3x_3 = 0, \\ 3x_1 + 3x_2 + (-3 - \lambda)x_3 = 0 \end{cases}$$

bir jinsli sistemaning nolmas yechimini topamiz:

$$\Delta(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 4 - \lambda & 3 & -3 \\ -3 & -2 - \lambda & 3 \\ 3 & 3 & -3 - \lambda \end{vmatrix} = 0.$$

Xarakteristik tenglamadan $\lambda_1 = -3$, $\lambda_2 = \lambda_3 = 1$ xos qiymatlarni topib olamiz. So'ngra ushbu

$$\begin{cases} 7x_1 + 3x_2 - 3x_3 = 0, \\ -3x_1 + x_2 + 3x_3 = 0, \\ 3x_1 + 3x_2 - 0 = 0 \end{cases}$$

sistemadan $\lambda_1 = -2$ xos qiymatga mos keluvchi $h_1 = (3, -3, 4)^T$ xos vektorni topamiz.

Ushbu $(A - \lambda_2 I)$ matritsaning rangi $r = r(A - \lambda_2 I) = 1$ bo'lgani uchun $\lambda = \lambda_2 = 1$ xos qiymatga mos keluvchi h_2, h_3 xos vektorlar chiziqli erkli bo'ladi, ya'ni

$$h_2 = (1, 0, 1)^T, \quad h_3 = (-1, 1, 0)^T.$$

Topilgan h_1, h_2, h_3 xos vektorlar R^3 fazoning bazisini tashkil qiladi. Shuning uchun bir jinsli sistemaning umumiy yechimi

$$y_u(x) = C_1 e^{-2x} h_1 + C_2 e^x h_2 + C_3 e^x h_3$$

ko'rinishida bo'ladi. Berilgan bir jinslimas sistemada $\mu = -1 \neq \lambda$ bo'lgani uchun, uning xususiy yechimini

$$\bar{y}(x) = e^{-x} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

ko'rinishda izlaymiz va $a = 3, b = -3, c = 2$ ekanligini topamiz. Shunday qilib, berilgan sistemaning umumiy yechimi

$$y(x) = y_u(x) + \bar{y}(x) = C_1 e^{-2x} h_1 + C_2 e^x h_2 + C_3 e^x h_3 + e^{-x} \begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix}$$

ko'rinishda bo'lar ekan. Bu yerda $C_j, j = \overline{1, 3}$ ixtiyoriyo'zgarmas sonlar.

Mustaqil yechish uchun mashqlar [8], §23, №810-824; [21], §14, №766-784.

6-§. Matritsaviy eksponenta

Faraz qilaylik, $x \in R$, A n o'lchamli kvadrat kompleks matritsa bo'lib, E n o'lchamli birlik matritsa bo'lsin.

Quyidagi

$$E + \frac{x}{1!} A + \frac{x^2}{2!} A^2 + \dots + \frac{x^k}{k!} A^k + \dots \quad (4.6.1)$$

matritsaviy darajali qatorni qaraylik. Agar

$$A = (a_{ij}), \quad i, j = \overline{1, n}, \quad A^2 = (a_{ij}^{(2)}), \quad i, j = \overline{1, n}, \dots, \quad A^k = (a_{ij}^{(k)}), \quad i, j = \overline{1, n}$$

ko‘rinishda bo‘lsa, u holda (4.6.1) matritsaviy darajali qatorning ixtiyoriy elementi

$$\delta_{i,j} + \frac{x}{1!} a_{ij} + \frac{x^2}{2!} a_{ij}^{(2)} + \dots + \frac{x^k}{k!} a_{ij}^{(k)} + \dots \quad (4.6.2)$$

ko‘rinishda bo‘ladi. Bu yerda $\delta_{i,j}$ – Kroneker simvoli.

4.6.1-ta’rif. Agar (4.6.2) darajali qator ixtiyoriy $i, j = \overline{1, n}$ va $x \in R$ da absolyut yaqinlashuvchi bo‘lsa, (4.6.1) matritsaviy darajali qator $x \in R$ da absolyut yaqinlashuvchi deyiladi.

4.6.1-lemma. Ixtiyoriy A matritsa va har bir $x \in R$ uchun (4.6.1) matritsaviy darajali qator absolyut yaqinlashuvchi bo‘ladi.

Isbot. Shunday $M > 0$ soni topilib, A matritsaning barcha elementlari uchun

$$|a_{ij}| \leq M, \quad \forall i, j = \overline{1, n}$$

bahoo‘rinli bo‘ladi. Agar $A^2 = A \cdot A$ ekanligini inobatga olsak, u holda

$$a_{ij}^{(2)} = \sum_{p=1}^n a_{ip} \cdot a_{pj}$$

munosabat o‘rinli bo‘ladi. Bundan

$$|a_{ij}^{(2)}| \leq \sum_{p=1}^n |a_{ip}| \cdot |a_{pj}| \leq nM^2, \quad i, j = \overline{1, n}$$

tengsizlik kelib chiqadi. Matematik induksiya usulini qo‘llab

$$|a_{ij}^{(k)}| \leq n^{k-1} M^k, \quad \forall k \in N; \quad i, j = \overline{1, n}$$

bahoni olish mumkin. Shuning uchun (4.6.2) darajali qatorning majarantasi ushbu

$$1 + \frac{|x|}{1!} M + \frac{|x|^2}{2!} nM^2 + \dots + \frac{|x|^k}{k!} n^{k-1} M^k + \dots$$

ko‘rinishni oladi. Koshi alomatiga ko‘ra, bu qator yaqinlashadi. Bundan (4.6.2) va (4.6.1) darajali qatorlarning har bir $x \in R$ larda absolyut yaqinlashishi kelib chiqadi. ■

4.6.2-ta’rif. Absolyut yaqinlashuvchi (4.6.1) darajali qatorning yig‘indisiga matritsaviy eksponenta deyiladi va

$$e^{xA} = E + \sum_{k=1}^{\infty} \frac{x^k}{k!} A^k \quad (4.6.3)$$

ko‘rinishda yoziladi.

4.6.1-izoh. Har bir $[a, b] \subset R$ kesmada (4.6.3) qator tekis yaqinlashadi.

Ko‘rinib turibdiki, agar $A=0$ yoki $A=E$ bo‘lsa, u holda

$$e^{x \cdot 0} = E, \quad e^{xE} = e^x \cdot E$$

munosabatlari o‘rinli bo‘ladi.

4.6.2-lemma. Agar bir xil o'ldamli A va B matritsalar uchun $AB=BA$ tenglik o'rinli bo'lsa, u holda

$$e^{xA} \cdot e^{xB} = e^{xB} \cdot e^{xA} = e^{x(A+B)}, \quad \forall x \in R$$

munosabat bajariladi.

Isbot. Ushbu $AB=BA$ tenglikdan va matematik induksiya usulidan foydalanib, quyidagi

$$(A+B)^n = \sum_{k=0}^n \frac{n!}{k!(n-k)!} A^k B^{n-k} = n! \sum_{k+m=n} \frac{A^k B^m}{k!m!}$$

binom formulasining o'rinli bo'lishini ko'rsatish mumkin. Bunga asoslanib, ushbu

$$\begin{aligned} e^{x(A+B)} &= \sum_{n=0}^{\infty} \frac{x^n}{n!} (A+B)^n = \sum_{n=0}^{\infty} \sum_{k+m=n} \frac{x^k A^k}{k!} \frac{x^m B^m}{m!} = \\ &= \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{x^k A^k}{k!} \frac{x^m B^m}{m!} = \sum_{k=0}^{\infty} \frac{x^k A^k}{k!} e^{xB} = e^{xA} e^{xB} = e^{xB} e^{xA} \end{aligned}$$

formulani keltirib chiqaramiz. Yuqoridagi tenglikni keltirib chiqarishda ikki karrali qatorning absolyut yaqinlashishi inobatga olindi.

Agar $B=-A$ bo'lsa, u holda 4.6.2-lemmadan e^{-xA} matritsa e^{xA} ning teskari matritsasidan iborat ekanligi kelib chiqadi. o'z navbatida e^{xA} ning xosmas matritsa ekanligiga ishonch hosil qilamiz.

Bundan tashqari, (4.6.2) darajali qatorni hadlab differentsiallash mumkinligidan foydalanib, e^{xA} matritsaning barcha tartibli hosilalarini hisoblash mumkin:

$$\frac{d}{dx} e^{xA} = e^{xA} A = A e^{xA}.$$

Haqiqatan ham,

$$\begin{aligned} \frac{d}{dx} e^{xA} &= \frac{d}{dx} \left(E + \frac{x}{1!} A + \frac{x^2}{2!} A^2 + \dots + \frac{x^k}{k!} A^k + \dots \right) = \\ &= A + \frac{x}{1!} A^2 + \dots + \frac{x^{k-1}}{(k-1)!} A^k + \dots = A e^{xA} = e^{xA} A. \end{aligned}$$

Ixtiyoriy A kvadrat matritsa uchun e^{xA} matritsaviy eksponentani hisoblash masalasi ancha murakkab masala hisoblanadi.

4.6.3-lemma. Agar

$$A = HBH^{-1}, \quad \det H \neq 0$$

ko'rinishda bo'lsa, u holda

$$e^{xA} = H e^{xB} H^{-1}, \quad \forall x \in R$$

tenglik bajariladi.

Isbot. Quyidagi munosabat o'rinli:

$$A^2 = A \cdot A = HBH^{-1} \cdot HBH^{-1} = HB^2H^{-1}.$$

Matematik induksiya usulini qo'llab,

$$A^k = HB^kH^{-1}, \quad \forall k \in \mathbb{N}$$

bo'lishini topamiz. Bundan va (4.6.3) qatordan foydalansak

$$e^{xA} = HH^{-1} + \sum_{k=1}^{\infty} \frac{x^k}{k!} HB^kH^{-1} = H \left(E + \sum_{k=1}^{\infty} \frac{x^k}{k!} B^k \right) H^{-1} = He^{xB}H^{-1}$$

kelib chiqadi. ■

Shunday qilib, A va e^{xA} matritsalar H xosmas almashtirish natijasida bir xil qonun bo'yicha o'zgarar ekan.

Chiziqli algebra kursidan ma'lumki, agar A matritsaning xos vektorlaridan tashkil topgan $\{h_1, h_2, \dots, h_n\}$ vektorlar sistemasi R^n fazoning bazisini tashkil qilsa, u holda $B = HAH^{-1}$ matritsa diagonal ko'rinishga keladi:

$$B = \begin{pmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & \dots & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot \\ 0 & 0 & 0 & \dots & \lambda_n \end{pmatrix}.$$

Bu yerda diagonal elementlar A matritsaning λ_j ($j = \overline{1, n}$) xos qiymatlaridan iborat bo'ladi. Bunda H matritsa ustun elementlari h_1, h_2, \dots, h_n vektorlarning R^n fazoning e_1, e_2, \dots, e_n bazisidagi koordinatalaridan iborat. Bu holda (4.6.3) formuladan ushbu

$$e^{xB} = \begin{pmatrix} e^{\lambda_1 x} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 x} & \dots & 0 \\ \cdot & \cdot & \dots & \cdot \\ 0 & 0 & \dots & e^{\lambda_n x} \end{pmatrix}$$

tenglikni topamiz va 4.6.3-lemmaga asosan

$$e^{xA} = He^{xB}H^{-1}$$

o'rinli. Haqiqatan ham, ushbu

$$B^n = \begin{pmatrix} \lambda_1^n & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & \lambda_n^n \end{pmatrix}$$

tenglikdan va (4.6.3) qatordan foydalanib, e^{xB} matritsani hisoblash mumkin:

$$\begin{aligned}
e^{xB} &= E + \frac{x}{1!}B + \frac{x^2}{2!}B^2 + \dots + \frac{x^n}{n!}B^n + \dots = \begin{vmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \cdot & \cdot & \dots & \cdot \\ 0 & 0 & \dots & 1 \end{vmatrix} + \\
&+ \begin{vmatrix} \frac{x}{1!}\lambda_1 & 0 & \dots & 0 \\ 0 & \frac{x}{1!}\lambda_2 & \dots & 0 \\ \cdot & \cdot & \dots & \cdot \\ 0 & 0 & \dots & \frac{x}{1!}\lambda_n \end{vmatrix} + \begin{vmatrix} \frac{x^2}{2!}\lambda_1^2 & 0 & \dots & 0 \\ 0 & \frac{x^2}{2!}\lambda_2^2 & \dots & 0 \\ \cdot & \cdot & \dots & \cdot \\ 0 & 0 & \dots & \frac{x^2}{2!}\lambda_n^2 \end{vmatrix} + \dots + \\
&+ \begin{vmatrix} \frac{x^n}{n!}\lambda_1^n & 0 & \dots & 0 \\ 0 & \frac{x^n}{n!}\lambda_2^n & \dots & 0 \\ \cdot & \cdot & \dots & \cdot \\ 0 & 0 & \dots & \frac{x^n}{n!}\lambda_n^n \end{vmatrix} + \dots = \\
&= \begin{vmatrix} 1 + \frac{x}{1!}\lambda_1 + \frac{x^2}{2!}\lambda_1^2 + \dots + \frac{x^n}{n!}\lambda_1^n & \dots & \dots & 0 \\ \cdot & \dots & \dots & \cdot \\ 0 & \dots & 1 + \frac{x}{1!}\lambda_1 + \frac{x^2}{2!}\lambda_1^2 + \dots + \frac{x^n}{n!}\lambda_1^n & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} = \\
&= \begin{vmatrix} e^{x\lambda_1} & 0 & \dots & 0 \\ 0 & e^{x\lambda_2} & \dots & 0 \\ \cdot & \cdot & \dots & \cdot \\ 0 & 0 & \dots & e^{x\lambda_n} \end{vmatrix}.
\end{aligned}$$

Chiziqli algebra kursidan ma'lumki, ko'pchilik hollarda A matritsaning xos vektorlaridan tashkil topgan $\{h_1, h_2, \dots, h_n\}$ vektorlar sistemasi R^n fazoning bazisini tashkil qilavermaydi. Ammo R^n fazoda (Jordan teoremasiga ko'ra), ixtiyoriy A matritsaning barcha xos qiymatlariga mos keluvchi Jordan zanjiridan tashkil topgan Jordan bazisi mavjud. A almashtirishning (chiziqli operatorning) bu bazisdagi matritsasini J orqali belgilaymiz. Bu J matritsaga A ning normal Jordan formasi deyiladi. Ma'lumki, A matritsaning k karrali λ xos qiymatiga mos keluvchi Jordan katagi k o'lchamli kvadrat matritsa bo'lib, u ushbu

$$J_k(\lambda) = \begin{vmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \dots & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & \lambda \end{vmatrix}$$

ko‘rinishda bo‘ladi. Bu yerda $k - J_k(\lambda)$ Jordan kataginingo‘lchami. Masalan,

$$J_1(\lambda) = \|\lambda\|, J_2(\lambda) = \begin{vmatrix} \lambda & 1 \\ 0 & \lambda \end{vmatrix}, J_3(\lambda) = \begin{vmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{vmatrix}.$$

Faraz qilaylik, A matritsaning k_1, k_2, \dots, k_m karrali $\lambda_1, \lambda_2, \dots, \lambda_m$ xos qiymatlariga ($k_1 + k_2 + \dots + k_m = n$) mos keluvchi m ta ($1 \leq m \leq n$) Jordan zanjiri R^n fazoning bazisini tashkil qilsin. U holda A matritsaning J Jordan formasi katakli–diagonal ko‘rinishga ega bo‘ladi:

$$J = \begin{vmatrix} J_{k_1}(\lambda_1) & 0 & \dots & 0 \\ 0 & J_{k_2}(\lambda_2) & \dots & 0 \\ \cdot & \cdot & \dots & 0 \\ 0 & 0 & \dots & J_{k_m}(\lambda_m) \end{vmatrix}.$$

Agar $k_1 = k_2 = \dots = k_m = 1$ bo‘lsa, u holda J diagonal matritsadan iborat bo‘ladi. Katakli – diagonal J matritsa qisqacha ushbu

$$J = \text{diag}\{J_{k_1}(\lambda_1), J_{k_2}(\lambda_2), \dots, J_{k_m}(\lambda_m)\}$$

ko‘rinishda belgilanadi.

Agar H orqali avvalgi e_1, e_2, \dots, e_n bazisdan Jordan bazisiga o‘tish matritsasini belgilasak, u holda

$$J = HAH^{-1}$$

tasviro‘rinli bo‘ladi. 4.6.3-lemmadan esa ushbu

$$e^{xA} = He^{xJ}H^{-1}$$

munosabat kelib chiqadi. o‘z navbatida (4.6.3) qatordan va katakli – diagonal matritsaning xossalaridan foydalanib

$$e^{xJ} = \text{diag}\{e^{xJ_{k_1}(\lambda_1)}, e^{xJ_{k_2}(\lambda_2)}, \dots, e^{xJ_{k_m}(\lambda_m)}\}$$

tenglikni hosil qilamiz. Shunday qilib, e^{xA} matritsani hisoblash masalasi $e^{xJ_{k_j}(\lambda_j)}$ matritsani hisoblash masalasiga keltiriladi.

Endi k – tartibli $J_k(0)$ matritsani qaraylik:

$$J_k(0) = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

U holda

$$xJ_k(\lambda) = x\lambda E_k + xJ_k(0), \quad E_k J_k(0) = J_k(0) E_k$$

munosabat bajariladi. Bunda E_k k o'lchamli birlik matritsa. 4.6.2-lemmadan foydalanib,

$$e^{xJ_k(\lambda)} = e^{x\lambda E_k} \cdot e^{xJ_k(0)} = e^{x\lambda} \cdot e^{xJ_k(0)}$$

munosabatni olamiz. Ushbu $e^{xJ_k(0)}$ matritsani (4.6.1) qatordan foydalanib hisoblash mumkin. Ko'rinib turibdiki,

$$J_k^m(0) = 0, \quad m \geq k.$$

Chunki

$$J_k^2(0) = \begin{pmatrix} 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ \cdot & \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}, \dots, J_k^{k-1}(0) = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 \\ \cdot & \cdot & \dots & \cdot & \cdot \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

Bu munosabatlardan foydalanib,

$$\begin{aligned} e^{xJ_k(0)} &= E_k + \frac{x}{1!} J_k(0) + \dots + \frac{x^{k-1}}{(k-1)!} J_k^{k-1}(0) = \\ &= \begin{pmatrix} 1 & x & \frac{x^2}{2!} & \dots & \frac{x^{k-1}}{(k-1)!} \\ 0 & 1 & \frac{x}{1!} & \dots & \frac{x^{k-2}}{(k-2)!} \\ \cdot & \cdot & \cdot & \dots & \cdot \\ 0 & 0 & 0 & \dots & \frac{x}{1!} \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \end{aligned}$$

tenglikni hosil qilamiz. Bundan

$$e^{xJ} = \text{diag}\{e^{\lambda_1 x} \cdot e^{xJ_{k_1}(0)}, \dots, e^{\lambda_k x} \cdot e^{xJ_{k_m}(0)}\}$$

kelib chiqadi. Bu yerda $J_{k_1}(0), J_{k_2}(0), \dots, J_{k_m}(0)$ -mos ravishda k_1, k_2, \dots, k_m o'lchamli kvadrat matritsalar.

4.6.1-misol. Ushbu

$$A = \begin{pmatrix} 1 & 4 \\ -1 & 5 \end{pmatrix}$$

matritsaning e^{-xA} , $x \in R$ matritsaviy eksponentasini hisoblang.

Yechish. Avvalo A matritsaning xos qiymatlarini va xos vektorlarini topib olamiz. Buning uchun $Ah = \lambda h$, $0 \neq h = (x_1, x_2)^T$ tenglamani qaraymiz:

$$\begin{cases} x_1 + 4x_2 = \lambda x_1, \\ -x_1 + 5x_2 = \lambda x_2 \end{cases} \Rightarrow \begin{cases} (1 - \lambda)x_1 + 4x_2 = 0, \\ -x_1 + (5 - \lambda)x_2 = 0 \end{cases}$$

$$\begin{vmatrix} 1 - \lambda & 4 \\ -1 & 5 - \lambda \end{vmatrix} = 0, \quad (1 - \lambda)(5 - \lambda) + 4 = 0,$$

$$5 - \lambda - 5\lambda + \lambda^2 + 4 = 0, \quad \lambda^2 - 6\lambda + 9 = 0.$$

Bu kvadrat tenglamadan $\lambda_1 = \lambda_2 = 3$ karrali xos qiymatni topamiz. Bu xos qiymatga $h_1 = (2, 1)^T$ xos vektor mos keladi. Endi h_1 vektorga yopishgan h_2 vektorni quyidagi

$$Ah_2 = 3h_2 + h_1$$

tenglamadan topamiz: $h_2 = (-1, 0)^T$.

Ushbu h_1, h_2 vektorlar R^2 fazoning Jordan bazisini tashkil qiladi. Quyidagi

$$H = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}, \quad H^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$$

xosmas matritsalarini tuzib,

$$J = H^{-1}AH = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$$

bo'lishini topamiz. Bundan

$$e^{xJ} = e^{3x} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \text{ va } e^{xA} = He^{xJ}H^{-1} = e^{3x} \begin{pmatrix} 1 - 2x & 4x \\ -x & 1 + 2x \end{pmatrix}$$

kelib chiqadi.

Yuqoridagi tushunchalardan tashqari quyidagi tasdiqningo'rinli ekanligini ham ko'rsatish mumkin.

4.6.1-teorema. Ushbu $Y(x) = e^{xA}$ matritsa funksiya quyidagi

$$\frac{dY(x)}{dx} = AY(x), \quad Y(0) = E$$

Koshi masalasining yechimidan iborat bo'ladi.

Bundan e^{xA} – matritsa ushbu

$$\frac{dy}{dx} = Ay, \quad y = (y_1, y_2, \dots, y_n)^T$$

differensial tenglamalar sistemasining fundamental matritsasidan iborat bo'lishi kelib chiqadi.

4.6.1-natija. Ushbu

$$\det e^{xA} = e^{xspA}$$

formulao‘rinli. Bu yerda

$$spA = \sum_{j=1}^n a_{jj}$$

A matritsaning izi.

4.6.2-misol. Agar

$$A = \begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix}$$

ko‘rinishdagi matritsa bo‘lsa, u holda e^{xA} – matritsani hisoblang.

Yechish. Avvalo A matritsaning xos qiymatlarini topamiz:

$$\Delta(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 4 - \lambda & -3 \\ 2 & -1 - \lambda \end{vmatrix} = 0, \quad \lambda^2 - 3\lambda + 2 = 0.$$

Bu kvadrat tenglamani yechib, $\lambda_1 = 1, \lambda_2 = 2$ xos qiymatlarni topamiz. Endi $\lambda_1 = 1$ xos qiymatga mos keluvchi xos vektorni topamiz:

$$\begin{vmatrix} 3 & -3 \\ 2 & -2 \end{vmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad x_1 = 1, \quad x_2 = 1,$$

ya’ni $h_1 = (1, 1)^T$ xos vektorni topamiz. Xuddi shuningdek,

$$\begin{vmatrix} 2 & -3 \\ 2 & -3 \end{vmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad x_1 = 3, \quad x_2 = 2,$$

ya’ni ikkinchi, $\lambda_2 = 2$ xos qiymatga mos keluvchi $h_2 = (3, 2)^T$ xos vektorni topamiz.

Bu ma’lumotlardan foydalanib,

$$\frac{dy}{dx} = Ay$$

tenglamaning umumiy yechimini aniqlaymiz:

$$y(x) = c_1 e^{\lambda_1 x} h_1 + c_2 e^{\lambda_2 x} h_2 = c_1 e^x \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{2x} \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Endi yuqoridagi differensial tenglamaning ushbu

$$y^{(1)}(x)|_{x=0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad y^{(2)}(x)|_{x=0} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

boshlang‘ich shartlarni qanoatlantiruvchi yechimlarini aniqlaymiz:

$$y^{(1)}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$
$$c_1 = -2, \quad c_2 = 1,$$

$$y^{(1)}(x) = -2e^x \begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{2x} \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

$$y^{(2)}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$c_1 = 3, \quad c_2 = -1,$$

$$y^{(2)}(x) = 3e^x \begin{pmatrix} 1 \\ 1 \end{pmatrix} - e^{2x} \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Topilgan $y^{(1)}(x)$ va $y^{(2)}(x)$ yechimlardan foydalanib, e^{xA} matritsani hisoblaymiz:

$$e^{-xA} = \begin{vmatrix} -2e^x + 3e^{2x} & 3e^x - 3e^{2x} \\ -2e^x + 2e^{2x} & 3e^x - 2e^{2x} \end{vmatrix}.$$

7-§. O'zgarmas koeffitsiyentli chiziqli bir jinsli bo'lmagan differensial tenglamalar sistemasini yechishda matritsaviy eksponentadan foydalanish

Ushbu

$$\frac{dy}{dx} = Ay + f(x), \quad x \in R, \quad (4.7.1)$$

$$y(x_0) = y^0 \quad (4.7.2)$$

Koshi masalasini qaraylik. Bu yerda $y = y(x) = (y_1(x), \dots, y_n(x))^T$ – noma'lum vektor-funksiya, A – n o'lchamli o'zgarmas kvadrat matritsa, $f(x) = (f_1(x), \dots, f_n(x))^T$ – berilgan uzluksiz vektor-funksiya, x_0 va y^0 berilgan n o'lchamli sonli vektorlar.

4.7.1-teorema. (4.7.1) differensial tenglamalar sistemasining umumiy yechimi uchun quyidagi

$$y(x) = e^{xA} \cdot C + e^{xA} \int_{x_0}^x e^{-\tau A} f(\tau) d\tau \quad (4.7.3)$$

formula o'rinli. Bu yerda $x_0, x \in [a, b] \subseteq R$, C – n o'lchamli ixtiyoriy sonli vektor.

Isbot. Ushbu

$$y(x) = e^{xA} z(x) \quad (4.7.4)$$

almashtirishdan foydalanib, (4.7.1) differensial tenglamalar sistemasini quyidagicha yozish mumkin. Bunda $z(x) = (z_1, \dots, z_n)^T$ – yangi noma'lum vektor-funksiya:

$$y'(x) = e^{xA} Az(x) + e^{xA} z'(x),$$

$$e^{xA} Az(x) + e^{xA} z'(x) = Ae^{xA} z(x) + f(x),$$

$$y'(x) = (y'_1(x), y'_2(x), \dots, y'_n(x))^T.$$

Bu belgilashlar yordamida (4.8.1) sistema ushbu

$$y'(x) = f(x, y) \quad (4.8.2)$$

ko'rinishni oladi. Bu yerda $y(x)$ – noma'lum vektor-funksiya, $f(x, y) – G \subset R^{n+1}$ sohada uzluksiz berilgan vektor-funksiya. Odatda bunday vektor-funksiyalardan tuzilgan fazoni qisqacha $f(x, y) \in C_n(G)$ deb belgilaymiz.

4.8.1-ta'rif. Agar biror $y = \varphi(x)$ – vektor-funksiya $I = (a, b) \subset R$ oraliqda quyidagi

1. $\varphi(x) – I$ oraliqda bir marta uzluksiz differensiallanuvchi vektor-funksiya, ya'ni $\varphi(x) \in C_n^1(I)$.

2. $(x, \varphi(x)) \in G, \forall x \in I$

3. $\varphi'(x) \equiv f(x, \varphi(x)), x \in I$

shartlarni qanoatlantirsa, unga (4.8.2) sistemaning yechimi deyiladi. Bu $y = \varphi(x), x \in I$ yechimning grafigiga, ya'ni $(x, \varphi(x)), x \in I$ ko'rinishdagi nuqtalar to'plamiga (4.8.2) sistemaning integral chizig'i deyiladi.

Endi (4.8.2) ko'rinishdagi differensial tenglamalar sistemasiga qo'yilgan

$$y(x_0) = y^0, (x_0, y^0) \in G, x_0 \in I \quad (4.8.3)$$

Koshi masalasini qaraymiz. Bu yerda $y^0 = (y_1^0, y_2^0, \dots, y_n^0)$ – berilgan sonli vektor, $x_0 \in I$ – berilgan son, $(x_0, y^0) \in G$ – boshlang'ich nuqta.

4.8.2-ta'rif. (4.8.2) differensial tenglamalar sistemasining (4.8.3) boshlang'ich shartni qanoatlantiruvchi yechimini topishga Koshi masalasi deyiladi.

Ushbu

$$y(x) = y^0 + \int_{x_0}^x f(\tau, y(\tau)) d\tau \quad (4.8.4)$$

ko'rinishdagi tenglamaga, integral tenglamalar sistemasi deyiladi. Bu yerda

$$(x_0, y^0) \in G, f(x, y) \in C_n(G), x_0 \in I, y^0 \in R^n.$$

Agar, biror $y = \varphi(x), x \in I$ vektor-funksiya quyidagi

1) $\varphi(x) \in C_n(I)$ – uzluksiz vektor-funksiya,

2) $(x, \varphi(x)) \in G, \forall x \in I,$

3) $\varphi(x) \equiv y^0 + \int_{x_0}^x f(\tau, \varphi(\tau)) d\tau, \forall x \in I$

shartlarni qanoatlantirsa, unga (4.8.4) integral tenglamalar sistemasining yechimi deyiladi.

Endi (4.8.2)-(4.8.3) Koshi masalasining yechimga egaligi (4.8.4) integral tenglamalar sistemasining yechimga egaligi masalasiga ekvivalent ekanligini ko'rsatamiz. Shu maqsadda quyidagi tasdiqni keltiramiz.

4.8.1-lemma. $y = \varphi(x), x \in I$ – vektor-funksiya (4.8.2)-(4.8.3) Koshi masalasining yechimi bo'lishi uchun u (4.8.4) integral tenglamalar sistemasining yechimi bo'lishi zarur va yetarli.

Bu lemmaning isbotini o'quvchiga havola qilamiz.

Berilgan (4.8.2)-(4.8.3) Koshi masalasida $(x_0, y^0) \in G \subseteq R^{n+1}$ bo'lib, G – soha ochiq to'plam bo'lgani uchun, shunday $\exists p > 0, q > 0$ sonlari topilib, (x_0, y^0) nuqtani o'z ichiga oluvchi chegaralangan yopiq $G_{pq} = \{(x, y) \in G : |x - x_0| \leq p, |y - y_0| \leq q\}$ – to'plam mavjud. Bunda $G_{pq} \subset G$.

4.8.1-teorema. Aytaylik, $f(x, y)$ vektor-funksiya G_{pq} to'plamda uzluksiz bo'lib, y o'zgaruvchi bo'yicha Lipshits shartini qanoatlantirsin. U holda:

1. Shunday $\exists \delta > 0$ soni topilib, $I = [x_0 - \delta, x_0 + \delta]$ oraliqda (4.8.2)-(4.8.3) Koshi masalasining yechimi mavjud bo'ladi.

2. Agar $y = \varphi(x), x \in I_1 = [x_0 - \delta_1, x_0 + \delta_1], \delta_1 > 0$ va $y = \psi(x), x \in I_2 = [x_0 - \delta_2, x_0 + \delta_2], \delta_2 > 0$ vektor-funksiyalar (4.8.2)-(4.8.3) Koshi masalasining yechimlari bo'lsa, u holda $\varphi(x) \equiv \psi(x), \forall x \in I_1 \cap I_2$ ayniyat bajariladi.

Isbot. Teorema shartiga ko'ra, $f(x, y) \in C_n(G_{pq})$ bo'lgani uchun, shunday $\exists M > 0$ soni topilib

$$|f(x, y)| \leq M, \forall (x, y) \in G_{pq}$$

o'rinli bo'ladi. Bundan tashqari $f(x, y)$ funksiya G_{pq} chegaralangan yopiq to'plamda y o'zgaruvchi bo'yicha Lipshits shartini qanoatlantirgani uchun, $\forall (x, y^1), (x, y^2) \in G_{pq}$ nuqtalar uchun shunday $\exists N > 0$ soni mavjud bo'lib,

$$|f(x, y^1) - f(x, y^2)| \leq N |y^1 - y^2|$$

tengsizlik o'rinli bo'ladi. Bu yerda N – Lipshits o'zgarishi bo'lib, u x ga bog'liq emas. Berilgan (4.8.2)-(4.8.3) Koshi masalasi (4.8.4) integral tenglamalar sistemasiga ekvivalent bo'lgani uchun, avvalo (4.8.4) sistemaning yechimini mavjudligini ko'rsatamiz. Buning uchun Pikarning ketma-ket yaqinlashishlar usulidan foydalanamiz.

Quyidagi

$$y_0(x) = y^0, y_{i+1}(x) = y^0 + \int_{x_0}^x f(\tau, y_i(\tau)) d\tau, x \in [x_0 - \delta, x_0 + \delta] \quad (4.8.7)$$

vektor-funksiyalar ketma-ketligini tuzib olamiz. Bu yerda

$$\begin{aligned}
y_0(x) &= (y_{10}(x), y_{20}(x), \dots, y_{n0}(x))^T, \\
y_i(x) &= (y_{1i}(x), y_{2i}(x), \dots, y_{ni}(x))^T, \\
y^0 &= (y_1^0, y_2^0, \dots, y_n^0)^T.
\end{aligned}$$

Avvalo $\forall x \in [x_0 - \delta, x_0 + \delta]$ lar uchun $(x, y_i(x)) \in G_{pq}$, $i = 0, 1, 2, 3, \dots$ ekanligini ko'rsatamiz. Buning uchun quyidagi ayirmani baholaymiz:

$$|y_i(x) - y_0(x)| = \left| \int_{x_0}^x f(\tau, y_0(\tau)) d\tau \right| \leq \int_{x_0}^x |f(\tau, y_0(\tau))| d\tau \leq M |x - x_0| \leq M \delta \leq q.$$

Chunki $\delta \leq q/M$. Bundan tashqari $f(x, y_0(x))$ vektor-funksiyaning uzluksizligidan va (4.8.7) tenglikdan $y_1(x)$ vektor-funksiyaning uzluksizligi kelib chiqadi. Demak, $(x, y_1(x)) \in G_{pq}$ ekan.

Faraz qilaylik, $[x_0 - \delta, x_0 + \delta]$ oraliqda $y_i(x)$ vektor-funksiya uzluksiz bo'lib, $(x, y_i(x)) \in G_{pq}$ bo'lsin. Matematik induksiya usulidan foydalanib $[x_0 - \delta, x_0 + \delta]$ oraliqda $y_{i+1}(x)$ vektor-funksiyaning uzluksizligini va $(x, y_{i+1}(x)) \in G_{pq}$ ekanligini ko'rsatish mumkin. Berilgan $f(x, y_i(x))$ vektor-funksiyaning $[x_0 - \delta, x_0 + \delta]$ oraliqda uzluksizligidan va (4.8.7) tenglikdan $y_{i+1}(x)$ vektor-funksiyaning $[x_0 - \delta, x_0 + \delta]$ oraliqda uzluksizligi kelib chiqadi. Bundan tashqari ushbu

$$|y_{i+1}(x) - y_0(x)| \leq \left| \int_{x_0}^x |f(\tau, y_0(\tau))| d\tau \right| \leq M |x - x_0| \leq M \delta \leq q.$$

munosabato'rinli. Shunday qilib $y_i(x)$, $i = 0, 1, 2, \dots$ vektor-funksiyalarning barchasi $[x_0 - \delta, x_0 + \delta]$ oraliqda uzluksiz bo'lib, ularning grafiklari $(x, y_i(x)) \in G_{pq}$ to'plamda yotar ekan. Endi $\{y_i(x)\}_{i=0}^{\infty}$ vektor-funksiyalar ketma-ketligining $[x_0 - \delta, x_0 + \delta]$ oraliqda $i \rightarrow \infty$ da tekis yaqinlashuvchiligini ko'rsatamiz. Buning uchun quyidagi

$$y_0(x) + \sum_{i=0}^{\infty} [y_{i+1}(x) - y_i(x)], \quad x \in [x_0 - \delta, x_0 + \delta] \quad (4.8.8)$$

qatorning tekis yaqinlashishini isbotlaymiz. Matematik induksiya usulidan foydalanib,

$$|y_{i+1}(x) - y_i(x)| \leq N^i M \frac{|x - x_0|^{i+1}}{(i+1)!}, \quad i = 0, 1, 2, \dots \quad (4.8.9)$$

bahoni olish mumkin. $i = 0$ holda bu bahoningo'rinli bo'lishi ko'rsatilgan edi. Aytaylik, ushbu

$$|y_i(x) - y_{i-1}(x)| \leq N^{i-1} M \frac{|x - x_0|^i}{i!}, \quad i = 0, 1, 2, \dots$$

bahoo‘rinli bo‘lsin. U holda (4.8.9) tengsizlikning bajarilishini ko‘rsatamiz:

$$\begin{aligned} |y_{i+1}(x) - y_i(x)| &\leq \left| \int_{x_0}^x |f(\tau, y_i(\tau)) - f(\tau, y_{i-1}(\tau))| d\tau \right| \leq N \left| \int_{x_0}^x |y_i(\tau) - y_{i-1}(\tau)| d\tau \right| \leq \\ &\leq N \frac{N^{i-1} M}{i!} \left| \int_{x_0}^x |\tau - x_0|^i d\tau \right| = N^i M \frac{|x - x_0|^{i+1}}{(i+1)!}. \end{aligned}$$

Chunki $x \in [x_0 - \delta, x_0 + \delta]$, $(x, y_i(x)) \in G_{pq}$. (4.8.9) bahodan

$$|y_{i+1}(x) - y_i(x)| \leq MN^i \frac{\delta^{i+1}}{(i+1)!}, \quad x \in [x_0 - \delta, x_0 + \delta]$$

tengsizlik kelib chiqadi. Bundan esa (4.8.8) qator yuqoridan ushbu

$$|y^0| + M \sum_{i=0}^{\infty} N^i \frac{\delta^{i+1}}{(i+1)!} = |y^0| + \frac{M}{N} (e^{\delta N} - 1)$$

yaqinlashuvchi sonli qator bilan baholanishi kelib chiqadi. Veyershtas alomatiga ko‘ra, bu (4.8.8) qator $[x_0 - \delta, x_0 + \delta]$ oraliqda biror $y = \varphi(x)$ vektor-funksiyaga tekis yaqinlashadi, ya‘ni $i \rightarrow \infty$ da

$$y_i(x) \rightarrow \varphi(x), \quad x \in [x_0 - \delta, x_0 + \delta].$$

Ushbu $[x_0 - \delta, x_0 + \delta]$ oraliqda tekis yaqinlashuvchi $\{y_i(x)\}$ vektor-funksiyalar ketma-ketligining limitik funksiyasi $\varphi(x)$ ham uzluksiz vektor-funksiyadan iborat bo‘ladi.

Quyidagi

$$|y_i(x) - y_0(x)| \leq q, \quad x \in [x_0 - \delta, x_0 + \delta]$$

tengsizlikda $i \rightarrow \infty$ da limitga o‘tib

$$|\varphi(x) - y_0| \leq q, \quad x \in [x_0 - \delta, x_0 + \delta]$$

bahoni olamiz. Bundan $(x, \varphi(x)) \in G_{pq}$ ekanligi kelib chiqadi.

Endi $y = \varphi(x)$, $x \in [x_0 - \delta, x_0 + \delta]$ vektor-funksiyaning (4.8.4) integral tenglamalar sistemasining yechimi ekanligini ko‘rsatamiz. Buning uchun $(x, \varphi(x)) \in G_{pq}$, $(x, y_i(x)) \in G_{pq}, i = 0, 1, 2, \dots$ munosabatlardan va (4.8.6) tengsizlikdan foydalanib

$$|f(x, y_i(x)) - f(x, \varphi(x))| \leq N |y_i(x) - \varphi(x)| \leq N \max_{[x_0 - \delta, x_0 + \delta]} |y_i(x) - \varphi(x)|,$$

$$x \in [x_0 - \delta, x_0 + \delta]$$

bahoni olamiz. Bu tengsizlikda $i \rightarrow \infty$ da limitga o‘tib,

$$f(x, y_i(x)) \rightarrow f(x, \varphi(x)), \quad x \in [x_0 - \delta, x_0 + \delta]$$

munosabatni topamiz. o‘z navbatida bu tengsizlikdan quyidagi

$$\int_{x_0}^x f(\tau, y_i(\tau))d\tau \rightarrow \int_{x_0}^x f(\tau, \varphi(\tau))d\tau, \quad x \in [x_0 - \delta, x_0 + \delta]$$

kelib chiqadi. Nihoyat, (4.8.7) tenglikda $i \rightarrow \infty$ da limitga o'tib,

$$\varphi(x) = y^0 + \int_{x_0}^x f(\tau, \varphi(\tau))d\tau, \quad x \in [x_0 - \delta, x_0 + \delta]$$

integral tenglamalar sistemasini hosil qilamiz. Bu esa $y = \varphi(x)$ vektor-funksiya $[x_0 - \delta, x_0 + \delta]$, $\delta = \min\left(p, \frac{q}{M}\right)$ oraliqda (4.8.4) integral tenglamalar sistemasining yechimi bo'lishini ko'rsatadi. Demak, 4.8.1-lemmaga asosan $y = \varphi(x)$ vektor-funksiya $[x_0 - \delta, x_0 + \delta]$ oraliqda (4.8.2)-(4.8.3) Koshi masalasining ham yechimi bo'lar ekan.

Endi yechimning yagonaligini isbotlaymiz.

Faraz qilaylik, $y = \varphi(x)$ vektor-funksiya (4.8.2)-(4.8.3) Koshi masalasining $I_1 = [x_0 - \delta_1, x_0 + \delta_1]$, $\delta_1 > 0$ oraliqdagi yechimi va $y = \psi(x)$ vektor-funksiya esa (4.8.2)-(4.8.3) Koshi masalasining $I_2 = [x_0 - \delta_2, x_0 + \delta_2]$, $\delta_2 > 0$ oraliqdagi yechimlari bo'lsin. U holda quyidagi

$$\varphi(x) = y^0 + \int_{x_0}^x f(\tau, \varphi(\tau))d\tau, \quad \forall x \in I_1,$$

$$\psi(x) = y^0 + \int_{x_0}^x f(\tau, \psi(\tau))d\tau, \quad \forall x \in I_2$$

integral tenglamalar sistemasio'rinli bo'ladi. Ushbu $x_0 \in \forall[\alpha, \beta] \subset I_1 \cap I_2$ oraliqda

$$|\varphi(x) - \psi(x)| \leq \left| \int_{x_0}^x |f(\tau, \varphi(\tau)) - f(\tau, \psi(\tau))|d\tau \right| \leq N \left| \int_{x_0}^x |\varphi(\tau) - \psi(\tau)|d\tau \right|,$$

$\forall x \in [\alpha, \beta]$ bahoo'rinli. Gronuolla tengsizligiga asosan $|\varphi(x) - \psi(x)| = 0$, $\forall x \in [\alpha, \beta]$ bajariladi. Bu esa $\varphi(x) = \psi(x)$, $\forall x \in [\alpha, \beta]$ ekanligini ko'rsatadi. ■

4.8.1-izoh. Ushbu

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)}),$$

$$y(x_0) = y_1^{(0)}, y'(x_0) = y_2^{(0)}, \dots, y^{(n-1)}(x_0) = y_n^{(0)}$$

ko'rinishdagi Koshi masalasini (4.8.1)-(4.8.3) ko'rinishdagi Koshi masalasiga keltirish mumkin. Haqiqatan ham $y = y_1, y' = y_2, \dots, y^{(n-1)} = y_n$ deb belgilashlar kritsak, n -tartibli differensial tenglama ushbu

$$\begin{cases} y_1' = y_2, \\ y_2' = y_3, \\ \dots\dots\dots \\ y_{n-1}' = y_n, \\ y_n' = f(x, y_1, y_2, \dots, y_n) \end{cases}$$

ko‘rinishdagi differensial tenglamalar sistemasiga keladi. Boshlang‘ich shartlar esa quyidagi

$$y_1(x_0) = y_1^{(0)}, y_2(x_0) = y_2^{(0)}, \dots, y_n(x_0) = y_n^{(0)}$$

ko‘rinishni oladi.

Shuning uchun n –tartibli hosilaga nisbatan yechilgan differensial tenglamaga qo‘yilgan Koshi masalasi (4.8.2)-(4.8.3) ko‘rinishdagi Koshi masalasining xususiy holdir.

9-§. O‘zgaruvchan koeffitsiyentli chiziqli bir jinsli bo‘lmagan differensial tenglamalar sistemasi uchun Koshi masalasi

Normal ko‘rinishdagi ushbu

$$\frac{dy_j}{dx} = \sum_{k=1}^n a_{jk}(x)y_k + f_j(x), \quad j = \overline{1, n}$$

chiziqli bir jinsli bo‘lmagan differensial tenglamalar sistemasi berilgan bo‘lsin. Bu yerda $y_j = y_j(x)$ – noma’lum, $a_{jk}(x)$ va $f_j(x)$ – ma’lum funksiyalar. Bu differensial tenglamalar sistemasini qulay ko‘rinishda yozish maqsadida quyidagi belgilashlarni kiritamiz:

$$\begin{aligned} y(x) &= (y_1(x), y_2(x), \dots, y_n(x))^T, \\ f(x) &= (f_1(x), f_2(x), \dots, f_n(x))^T, \\ A(x) &= \begin{vmatrix} a_{11}(x) & a_{12}(x) & \dots & a_{1n}(x) \\ a_{21}(x) & a_{22}(x) & \dots & a_{2n}(x) \\ \dots & \dots & \dots & \dots \\ a_{n1}(x) & a_{n2}(x) & \dots & a_{nn}(x) \end{vmatrix}, \\ \frac{dy}{dx} &= \left(\frac{dy_1}{dx}, \frac{dy_2}{dx}, \dots, \frac{dy_n}{dx} \right)^T. \end{aligned}$$

Bu belgilashlar yordamida berilgan differensial tenglamalar sistemasini ushbu

$$\frac{dy}{dx} = A(x)y + f(x) \tag{4.9.1}$$

ko‘rinishda yozish mumkin. (4.9.1) differensial tenglamalar sistemasining ushbu

$$y(x_0) = y^0 \tag{4.9.2}$$

boshlang'ich shartni qanoatlantiruvchi $y = y(x)$ yechimini topishga Koshi masalasi deyiladi. Bunda

$$y^0 = (y_1^0, y_2^0, \dots, y_n^0)$$

berilgan sonli vektor, $x_0 \in I = [a, b] \subset R$ – berilgan son.

4.9.1-teorema. Aytaylik, $f(x)$ vektor-funksiya va $A(x)$ – matritsa $I = [a, b]$ oraliqda uzluksiz hamda $x_0 \in I$, y^0 esa ixtiyoriy sonli vektor bo'lsin. U holda (4.9.1)-(4.9.2) Koshi masalasining I oraliqda aniqlangan yagona yechimi mavjud bo'ladi.

Isbot. Berilgan (4.9.1)-(4.9.2) Koshi masalasi ushbu

$$y(x) = y^0 + \int_{x_0}^x [A(\tau)y(\tau) + f(\tau)]d\tau \quad (4.9.3)$$

integral tenglamalar sistemasiga ekvivalent ekanligi oldingi paragraflardan ma'lum. Shuning uchun (4.9.3) integral tenglamalar sistemasining yechimini mavjudligini ko'rsatamiz. Shu maqsadda Pikarning ketma-ket yaqinlashishlar usulidan foydalanamiz. Faraz qilaylik, $x, x_0 \in [a, b]$, $x_0 \neq x$ bo'lsin. U holda quyidagi

$$y_0(x) = y^0, \quad (4.9.4)$$

$$y_{i+1}(x) = y^0 + \int_{x_0}^x [A(\tau)y_i(\tau) + f(\tau)]d\tau, \quad i = 0, 1, 2, \dots$$

$\{y_i(x)\}_{i=0}^{\infty}$ – ketma-ketliklarni tuzib olamiz. Berilgan $A(x)$, $f(x)$ funksiyalarning $[a, b]$ oraliqda uzluksizligidan va (4.9.4) tengliklardan $y_i(x)$, ($i = 0, 1, 2, \dots$) yaqinlashishlarning har biri $[a, b]$ oraliqda uzluksiz ekanligi kelib chiqadi. Endi (4.9.4) tengliklar yordamida aniqlangan $\{y_i(x)\}_{i=0}^{\infty}$ – ketma-ketlikning $[a, b]$ oraliqda tekis yaqinlashishini isbotlaymiz. Buning uchun ushbu

$$y_0(x) + \sum_{i=0}^{\infty} [y_{i+1}(x) - y_i(x)] \quad (4.9.5)$$

qatorning I oraliqda tekis yaqinlashishini ko'rsatamiz. Bu qatorning k – xususiy yig'indisi uchun

$S_k(x) = y_0(x) + y_1(x) - y_0(x) + y_2(x) - y_1(x) + \dots + y_k(x) - y_{k-1}(x) = y_k(x)$ munosabato'rinli.

Aytaylik, $A(x) = \|a_{ij}(x)\|$, $i, j = \overline{1, n}$ kvadrat matritsaning normasi

$$\|A(x)\| = \sqrt{\sum_{i=1}^n \sum_{j=1}^n |a_{ij}(x)|^2}, \quad x \in I = [a, b]$$

tenglik orqali aniqlangan bo'lsin. U holda quyidagi

$$\begin{aligned}\|\lambda A(x)\| &= |\lambda| \cdot \|A(x)\|, \quad \forall \lambda \in \mathbb{C}, \\ \|A_1(x) + A_2(x)\| &\leq \|A_1(x)\| + \|A_2(x)\|, \\ \|A_1(x) \cdot A_2(x)\| &\leq \|A_1(x)\| \cdot \|A_2(x)\|\end{aligned}$$

munosabatlarning o‘rinli ekanligini ko‘rsatish qiyinchilik tug‘dirmadi. Bunda

$$A_1(x) = \|a_{ij}^{(1)}(x)\|, A_2(x) = \|a_{ij}^{(2)}(x)\|, \quad i, j = \overline{1, n}.$$

$$\|A(x)y(x)\| \leq \|A(x)\| \cdot |y(x)|$$

tengsizlik ham o‘rinli. Bu yerda

$$y(x) = (y_1(x), y_2(x), \dots, y_n(x))^T, \quad |y(x)| = \sqrt{y_1^2(x) + \dots + y_n^2(x)}.$$

Berilgan $A(x)$ va $f(x)$ funksiyalarning $[a, b]$ oraliqda uzluksizligidan, shunday $K > 0$ va $M > 0$ sonlari topilib

$$|a_{ij}(x)| \leq K, \quad |f(x)| \leq M, \quad \forall x \in [a, b]$$

tengsizliklar bajariladi. Bundan

$$\|A(x)\| \leq nK, \quad \forall x \in [a, b]$$

baho kelib chiqadi.

Endi quyidagi ayirmani baholaymiz:

$$|y_1(x) - y_0(x)| \leq \left| \int_{x_0}^x [A(\tau)y_0 + f(\tau)] d\tau \right| \leq (nK|y_0| + M)(b - a) = C.$$

Bunda ushbu

$$\|A(x)y_0\| \leq \|A(x)\| \cdot |y_0| \leq nK|y_0|, \quad \forall x \in [a, b]$$

bahodan ham foydalandik.

Matematik induksiya usulini qo‘llab, quyidagi

$$|y_{i+1}(x) - y_i(x)| \leq C(nK)^i \frac{|x - x_0|^i}{i!}, \quad \forall x \in [a, b] \quad (4.9.6)$$

bahoni olish mumkin. $i=0$ bo‘lganda bu tengsizlikning o‘rinli ekanligi yuqorida ko‘rsatildi. Aytaylik (4.9.6) baho $i=m-1$ bo‘lganda o‘rinli bo‘lsin deb, uni $i=m$ uchun bajarilishini ko‘rsatamiz. Ushbu

$$\begin{aligned}|A(x)[y_m(x) - y_{m-1}(x)]| &\leq \|A(x)\| \cdot |y_m(x) - y_{m-1}(x)| \leq \\ &\leq nK|y_m(x) - y_{m-1}(x)|, \quad \forall x \in [a, b]\end{aligned}$$

tengsizlikdan foydalanib quyidagi bahoni olamiz:

$$\begin{aligned}|y_{m+1}(x) - y_m(x)| &\leq \left| \int_{x_0}^x |A(\tau)[y_m(\tau) - y_{m-1}(\tau)]| d\tau \right| \leq nK \left| \int_{x_0}^x |y_m(\tau) - y_{m-1}(\tau)| d\tau \right| \leq \\ &\leq nK \cdot C(nK)^{m-1} \left| \int_{x_0}^x \frac{|\tau - x_0|^{m-1}}{(m-1)!} d\tau \right| \leq C(nK)^m \frac{|x - x_0|^m}{m!}, \quad \forall x \in [a, b].\end{aligned}$$

O‘z navbatida (4.9.6) tengsizlikdan ushbu

$$|y_{i+1}(x) - y_i(x)| \leq C(nK)^i \frac{|b-a|^i}{i!}, \quad i = 0, 1, 2, \dots$$

baho ham kelib chiqadi.

Quyidagi

$$\left| y_0(x) + \sum_{i=0}^{\infty} [y_{i+1}(x) - y_i(x)] \right| \leq |y_0| + C \sum_{i=0}^{\infty} (nK)^i \frac{(b-a)^i}{i!} = |y_0| + Ce^{nK(b-a)}$$

tengsizlikdan Veyershtrass alomatiga ko'ra, (4.9.5) funksional qatorning $[a, b]$ oraliqda tekis yaqinlashishi kelib chiqadi. Bu esa (4.9.5) qatorning k -xususiy yig'indisining $k \rightarrow \infty$ da

$$S_k(x) \rightarrow \varphi(x), \quad x \in [a, b]$$

tekis yaqinlashishini bildiradi. Bu yerda

$$\varphi(x) = y_0(x) + \sum_{i=0}^{\infty} [y_{i+1}(x) - y_i(x)].$$

Demak, $i \rightarrow \infty$ da

$$y_i(x) \rightarrow \varphi(x), \quad x \in [a, b]$$

tekis yaqinlashadi. $[a, b]$ oraliqda tekis yaqinlashuvchi funksional ketma-ketlikning limitik funksiyasining uzluksizligidan $\varphi(x)$ ning $[a, b]$ da uzluksizligi kelib chiqadi. Bundan tashqari ushbu

$$|A(x)[y_i(x) - \varphi(x)]| \leq nK|y_i(x) - \varphi(x)|$$

tengsizlikdan va $\{y_i(x)\}$ ketma-ketlikning tekis yaqinlashuvchiligidan quyidagi

$$A(x)y_i(x) \rightarrow A(x)\varphi(x), \quad i \rightarrow \infty, x \in [a, b]$$

munosabat kelib chiqadi. Bu mulohazalar asosida (4.9.4) tenglikda $i \rightarrow \infty$ da limitga o'tsak

$$\varphi(x) = y^0 + \int_{x_0}^x [A(\tau)\varphi(\tau) + f(\tau)]d\tau$$

integral tenglamalar sistemasi kelib chiqadi. Demak, (4.9.4) tengliklar orqali aniqlangan $\{y_i(x)\}_{i=0}^{\infty}$ - Pikar ketma-ketligining $i \rightarrow \infty$ dagi limitik funksiyasi $\varphi(x), x \in [a, b]$ (4.9.3) integral tenglamalar sistemasining yechimidan iborat ekan. (4.9.3) integral tenglamalar sistemasi (4.9.1)-(4.9.2) Koshi masalasiga ekvivalent bo'lgani uchun, $\varphi(x), x \in [a, b]$ funksiya berilgan Koshi masalasining yechimidan iborat bo'ladi.

Endi yechimning yagonaligini isbotlaymiz.

Faraz qilaylik, (4.9.1)-(4.9.2) Koshi masalasi yechimi ikkita $y(x)$ va $z(x)$ vektor-funksiyalardan iborat bo'lsin:

$$\frac{dy}{dx} = A(x)y + f(x), \quad y(x_0) = y_0, \quad (4.9.7)$$

$$\frac{dz}{dx} = A(x)z + f(x), \quad z(x_0) = y_0. \quad (4.9.8)$$

Bu yerda

$$y(x) = (y_1(x), y_2(x), \dots, y_n(x))^T, \quad z(x) = (z_1(x), z_2(x), \dots, z_n(x))^T, \\ y^0 = (y_1^0, y_2^0, \dots, y_n^0), \quad x_0 \in I = [a, b], x \in [a, b].$$

Bu Koshi masalalari mos ravishda ushbu

$$y(x) = y^0 + \int_{x_0}^x [A(\tau)y(\tau) + f(\tau)]d\tau, \quad (4.9.9)$$

$$z(x) = y^0 + \int_{x_0}^x [A(\tau)z(\tau) + f(\tau)]d\tau \quad (4.9.10)$$

integral tenglamalar sistemasiga ekvivalent. Bu integral tenglamalar sistemasi uchun quyidagi Pikar yaqinlashishlarini qurish mumkin:

$$y_0(x) = y^0, \quad y_{i+1}(x) = y^0 + \int_{x_0}^x [A(\tau)y_i(\tau) + f(\tau)]d\tau, \quad (4.9.11)$$

$$z_0(x) = y^0, \quad z_{i+1}(x) = y^0 + \int_{x_0}^x [A(\tau)z_i(\tau) + f(\tau)]d\tau, \quad i = 0, 1, 2, \dots \quad (4.9.12)$$

Endi (4.9.10) tenglikdan (4.9.11) yaqinlashishni ayirib,

$$z(x) - y_{i+1}(x) = \int_{x_0}^x A(\tau)[z(\tau) - y_i(\tau)]d\tau \quad (4.9.13)$$

munosabatni hosil qilamiz va uni baholaymiz:

$$\begin{aligned} |z(x) - y_{i+1}(x)| &= \left| \int_{x_0}^x |A(\tau)[z(\tau) - y_i(\tau)]|d\tau \right| \leq \\ &\leq \left| \int_{x_0}^x \|A(\tau)\| \cdot |z(\tau) - y_i(\tau)|d\tau \right| \leq nK \left| \int_{x_0}^x |z(\tau) - y_i(\tau)|d\tau \right|, \end{aligned}$$

ya'ni

$$|z(x) - y_{i+1}(x)| \leq nK \left| \int_{x_0}^x |z(\tau) - y_i(\tau)|d\tau \right|. \quad (4.9.14)$$

Avvalo $i = 0$ bo'lganda ushbu $|z(x) - y_0(x)|$ ayirmanini baholaymiz:

$$|z(x) - y_0(x)| = |z(x) - y^0| \leq \left| \int_{x_0}^x |A(\tau)z(\tau) + f(\tau)|d\tau \right| \leq \left| \int_{x_0}^x |A(\tau)z(\tau)|d\tau \right| +$$

$$\begin{aligned}
& + \left| \int_{x_0}^x |f(\tau)| d\tau \right| \leq \left| \int_{x_0}^x \|A(\tau)\| \cdot |z(\tau)| d\tau \right| + M|x - x_0| \leq nK \left| \int_{x_0}^x |z(\tau)| d\tau \right| + \\
& + M|x - x_0| \leq nK \max_{x \in I} |z(x)| \cdot |x - x_0| + M|x - x_0| = (nKC + M)|x - x_0|.
\end{aligned}$$

$i = 1$ bo'lganda ushbu $|z(x) - y_1(x)|$ ayirmani baholaymiz:

$$\begin{aligned}
|z(x) - y_1(x)| & \leq \left| \int_{x_0}^x |A(\tau)(z(\tau) - y_0(\tau))| d\tau \right| \leq \\
& \leq \left| \int_{x_0}^x \|A(\tau)\| \cdot |z(\tau) - y_0(\tau)| d\tau \right| \leq nK \left| \int_{x_0}^x |z(\tau) - y_0(\tau)| d\tau \right| \leq \\
& \leq nK(nKC + M) \left| \int_{x_0}^x |\tau - x_0| d\tau \right| \leq nK(nKC + M) \frac{(x - x_0)^2}{2!}.
\end{aligned}$$

Xuddi shuningdek, $i = 2$ holida $|z(x) - y_2(x)|$ ayirmani baholaymiz:

$$\begin{aligned}
|z(x) - y_2(x)| & \leq \left| \int_{x_0}^x |A(\tau)(z(\tau) - y_1(\tau))| d\tau \right| \leq nK \left| \int_{x_0}^x |z(\tau) - y_1(\tau)| d\tau \right| \leq \\
& \leq (nK)^2(nKC + M) \left| \int_{x_0}^x \frac{(\tau - x_0)^2}{2!} d\tau \right| = (nKC + M) \frac{(nK)^2 |x - x_0|^3}{3!},
\end{aligned}$$

ya'ni

$$|z(x) - y_2(x)| \leq (nKC + M) \frac{(nK)^2 |x - x_0|^3}{3!}.$$

Matematik induksiya usulini qo'llab,

$$\begin{aligned}
|z(x) - y_i(x)| & \leq (nKC + M)(nK)^i \frac{|x - x_0|^{i+1}}{(i+1)!} \leq \\
& \leq (nKC + M)(nK)^i \frac{|b - a|^{i+1}}{(i+1)!}, \quad i = 0, 1, 2, \dots
\end{aligned}$$

tengsizlikning o'rinli ekanligini ko'rsatish mumkin. Oxirgi tengsizlikda $i \rightarrow \infty$ da limitga o'tsak,

$$|z(x) - y(x)| \leq 0$$

hosil bo'ladi. Bundan esa $y(x) \equiv z(x)$, $\forall x \in [a, b]$ kelib chiqadi. ■

4.9.1-natija. Ushbu

$$\frac{dy}{dx} = A(x)y(x), \quad y(x_0) = 0$$

Koshi masalasi faqat $y(x) = 0, x \in [a, b]$ yechimga ega. Bu yerda $A(x), [a, b]$ oraliqda aniqlangan uzluksiz matritsa funksiya, $x_0 \in [a, b]$ berilgan son.

Isbot. Ko‘rinib turibdiki, $y(x) \equiv 0, x \in [a, b]$ vektor-funksiya berilgan Koshi masalasining yechimidan iborat. Yagonalik teoremasiga ko‘ra bu yagona yechim.

4.9.1-izoh. Agar (4.9.1) sistemada $A = (a_{ij}), a_{ij} = \text{const}, i, j = \overline{1, n}$ – o‘zgarmas matritsa bo‘lib, $f(x) \equiv 0$ bo‘lsa, u holda ushbu

$$\frac{dy}{dx} = Ay, y(x_0) = y^0$$

Koshi masalasining yechimi

$$y(x) = e^{(x-x_0)A} \cdot y^0$$

ko‘rinishda tasvirlanadi.

10-§. Chiziqli bir jinsli differensial tenglamalar sistemasi. Chiziqli bog‘langan va bog‘lanmagan vektor-funksiyalar

Quyidagi

$$\frac{dy}{dx} = A(x)y \quad (4.10.1)$$

bir jinsli differensial tenglamalar sistemasi berilgan bo‘lsin. Bu yerda $A(x), x \in [a, b]$ oraliqda berilgan uzluksiz n o‘lchamli kvadrat matritsa-funksiya, $y(x) = (y_1(x), y_2(x), \dots, y_n(x))^T$ – noma’lum vektor-funksiya.

4.10.1-lemma. Agar $y^1(x), y^2(x)$ – vektor-funksiyalar (4.10.1) sistemaning yechimlaridan iborat bo‘lsa, u holda $y(x) = c_1 y^1(x) + c_2 y^2(x)$ vektor-funksiya ham (4.10.1) sistemaning yechimi bo‘ladi.

4.10.1-ta’rif. Agar kamida bittasi noldan farqli bo‘lgan c_1, c_2, \dots, c_k o‘zgarmas sonlar topilib, ushbu

$$c_1 y^1(x) + c_2 y^2(x) + \dots + c_k y^k(x) = 0, \forall x \in [a, b] \quad (4.10.2)$$

munosabato‘rinli bo‘lsa, $y^1(x), y^2(x), \dots, y^k(x)$ vektor-funksiyalar $[a, b]$ oraliqda chiziqli bog‘langan deyiladi. Aks holda, ya’ni (4.10.2) munosabat faqat $c_1 = c_2 = \dots = c_k = 0$ bo‘lganda bajarilsa, bu vektor-funksiyalarga chiziqli bog‘lanmagan deyiladi. Bu yerda

$$y^j(x) = (y_1^j(x), y_2^j(x), \dots, y_n^j(x))^T, j = \overline{1, k}.$$

4.10.2-lemma. Aytaylik, $\lambda_1, \lambda_2, \dots, \lambda_k$ sonlar berilgan bo‘lib, h_1, h_2, \dots, h_k chiziqli bog‘lanmagan vektorlar bo‘lsin. U holda $e^{\lambda_1 x} h_1, e^{\lambda_2 x} h_2, \dots, e^{\lambda_k x} h_k$ vektorlar chiziqli bog‘lanmagan vektorlar bo‘ladi.

Isbot. Teskarisini faraz qilamiz, u holda shunday c_1, c_2, \dots, c_k $|c_1| + |c_2| + \dots + |c_k| > 0$ sonlar mavjud bo‘lib,

$$c_1 e^{\lambda_1 x} h_1 + c_2 e^{\lambda_2 x} h_2 + \dots + c_k e^{\lambda_k x} h_k \equiv 0, \forall x \in R$$

munosabat bajariladi. Tayinlangan $x = x_0$ da h_1, h_2, \dots, h_k vektorlarning chiziqli bog‘lanmaganligidan $c_1 = c_2 = \dots = c_k = 0$ kelib chiqadi. Bu ziddiyat farazimizning noto‘g‘riligini bildiradi. ■

4.10.3-lemma. Agar $y^1(x), y^2(x), \dots, y^k(x)$ vektor-funksiyalar $[a, b]$ oraliqda chiziqli bog‘liq bo‘lsa, u holda $\forall x_0 \in [a, b]$ da $y^1(x_0), y^2(x_0), \dots, y^k(x_0)$ sonli vektorlar chiziqli bog‘liq bo‘ladi.

4.10.1-misol. Ushbu

$$y^1(x) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, y^2(x) = \begin{pmatrix} x \\ x \end{pmatrix}$$

vektor-funksiyalar R da chiziqli bog‘lanmagan. Lekin, $\forall x_0 \in R$ da $y^1(x_0), y^2(x_0)$ sonli vektorlar chiziqli bog‘langan, ya‘ni $y^2(x_0) = x_0 y^1(x_0)$.

4.10.3-lemmaning isbotini o‘quvchiga havola qilamiz.

4.10.1-teorema. Aytaylik, $y^j(x), j = \overline{1, k}$ vektor-funksiyalar (4.10.1) bir jinsli differensial tenglamalar sistemasining yechimlari bo‘lsin. U holda $y^j(x), j = \overline{1, k}$ yechimlarning $[a, b] \subseteq R$ oraliqda chiziqli bog‘liq bo‘lishi uchun, ushbu $\forall x_0 \in [a, b], y^j(x_0), j = \overline{1, k}$ sonli vektorlarning chiziqli bog‘liq bo‘lishi zarur va yetarli.

Isbot. Faraz qilaylik, (4.10.1) sistemaning $y^j(x), j = \overline{1, k}$ yechimlari chiziqli bog‘lanmagan bo‘lsin. Agar $\exists x_0 \in [a, b]$ mavjud bo‘lib, $y^j(x_0), j = \overline{1, k}$ sonli vektorlarning chiziqli bog‘liq bo‘lsa, u holda shunday c_1, c_2, \dots, c_k $|c_1| + |c_2| + \dots + |c_k| > 0$ sonlar topilib

$$c_1 y^1(x_0) + c_2 y^2(x_0) + \dots + c_k y^k(x_0) = 0$$

bajariladi. Ko‘rinib turibdiki, ushbu

$$y(x) = c_1 y^1(x) + c_2 y^2(x) + \dots + c_k y^k(x)$$

vektor-funksiya (4.10.1) sistemani va

$$y(x_0) = 0$$

boshlang‘ich shartni qanoatlantiradi. Bundan $y(x) \equiv 0, x \in [a, b]$ kelib chiqadi, ya‘ni $y^j(x), j = \overline{1, k}$ chiziqli bog‘liq vektor-funksiyalar bo‘lar ekan. Bu ziddiyat, farazimizning noto‘g‘riligini ko‘rsatadi.

Aksincha, faraz qilaylik, $\forall x_0 \in [a, b]$ da, ushbu $y^j(x_0), j = \overline{1, k}$ sonli vektorlar chiziqli bog‘lanmagan bo‘lsin. Agar bu $y^j(x), j = \overline{1, k}$ vektor-

funksiyalar $[a, b]$ da chiziqli bog‘liq bo‘lsa, u holda $y^j(x_0), j = \overline{1, k}$ sonli vektorlar ham chiziqli bog‘liq bo‘ladi. Bu esa ziddiyat. ■

4.10.1-natija. (4.10.1) bir jinsli sistemaning $y^j(x), j = \overline{1, k}$ yechimlari $[a, b]$ oraliqda chiziqli bog‘liq bo‘lishi uchun $\forall x_0 \in [a, b]$ da, ushbu $y^j(x_0), j = \overline{1, k}$ sonli vektorlarning chiziqli bog‘liq bo‘lishi zarur va yetarli.

4.10.2-ta’rif. Uzluksiz $y^j(x) \in C_n([a, b]), j = \overline{1, n}$ vektor-funksiyalardan tuzilgan ushbu

$$W(x) = W\{y^1(x), y^2(x), \dots, y^n(x)\} = \begin{vmatrix} y_1^1(x) & y_1^2(x) & \dots & y_1^n(x) \\ y_2^1(x) & y_2^2(x) & \dots & y_2^n(x) \\ \cdot & \cdot & \dots & \cdot \\ y_n^1(x) & y_n^2(x) & \dots & y_n^n(x) \end{vmatrix}$$

determinantga $\{y^1(x), y^2(x), \dots, y^n(x)\}$ vektor-funksiyalarning Vronskiy determinanti deyiladi.

4.10.4-lemma. Agar $y^1(x), y^2(x), \dots, y^n(x)$ vektor-funksiyalardan tuzilgan $W\{y^1(x), y^2(x), \dots, y^n(x)\} = W(x)$ Vronskiy determinanti biror $x_0 \in I = [a, b]$ nuqtada noldan farqli, ya’ni

$$W(x_0) \neq 0, x_0 \in [a, b]$$

bo‘lsa, u holda bu vektor-funksiyalar chiziqli bog‘lanmagan bo‘ladi.

Isbot. Teskarisini faraz qilaylik. Aytaylik, $y^j(x), j = \overline{1, n}$ vektor-funksiyalar chiziqli bog‘langan bo‘lsin. U holda kamida bittasi noldan farqli bo‘lgan c_1, c_2, \dots, c_n sonlar topilib

$$c_1 y^1(x) + c_2 y^2(x) + \dots + c_n y^n(x) = 0, x \in I = [a, b]$$

munosabato‘rinli bo‘ladi. Xususan, $x = x_0 \in [a, b]$ nuqtada

$$c_1 y^1(x_0) + c_2 y^2(x_0) + \dots + c_n y^n(x_0) = 0$$

bajariladi. $W(x_0) \neq 0$ bo‘lgani uchun $y^1(x_0), y^2(x_0), \dots, y^n(x_0)$ sonli vektorlar chiziqli bog‘lanmagandir. Shuning uchun oxirgi tenglikdan $c_1 = c_2 = \dots = c_n = 0$ kelib chiqadi. Bu qarama-qarshilik farazimizning noto‘g‘riligini ko‘rsatadi. ■

4.10.5-lemma. Agar $y^1(x), y^2(x), \dots, y^n(x)$ vektor-funksiyalar chiziqli bo‘langan bo‘lsa, u holda

$$W(x) = W\{y^1(x), y^2(x), \dots, y^n(x)\} \equiv 0, \forall x \in [a, b]$$

o‘rinli bo‘ladi.

Lemmaning isbotini o‘quvchiga havola qilamiz.

4.10.6-lemma. Agar $y^1(x), y^2(x), \dots, y^n(x)$ vektor-funksiyalar (4.10.1) differensial tenglamalar sistemasining yechimi bo‘lib, biror $x_0 \in I = [a, b]$ nuqtada

$$W(x_0) = W\{y^1(x_0), y^2(x_0), \dots, y^n(x_0)\} = 0$$

bo'lsa, u holda $y^j(x), j = \overline{1, n}$ vektor-funksiyalar $I = [a, b]$ oraliqda chiziqli bog'langan bo'ladi.

Isbot. Lemmaning shartiga ko'ra $W(x_0) = 0$ bo'lgani uchun, kamida bittasi noldan farqli bo'lgan $\exists c_1, c_2, \dots, c_n$ o'zgarmas sonlar topilib

$$c_1 y^1(x_0) + c_2 y^2(x_0) + \dots + c_n y^n(x_0) = 0$$

tenglik bajariladi. Ushbu

$$y(x) = c_1 y^1(x) + c_2 y^2(x) + \dots + c_n y^n(x) \quad (4.10.3)$$

vektor-funksiya (4.10.1) sistemani va

$$y(x_0) = 0 \quad (4.10.4)$$

boshlang'ich shartni qanoatlantiradi. Bundan tashqari $\bar{y}(x) = 0$ vektor-funksiya ham (4.10.1) sistemani va (4.10.4) boshlang'ich shartni qanoatlantiradi. Yagonalik teoremasiga ko'ra $y(x) = \bar{y}(x) = 0$ bo'ladi. Bunga asosan (4.10.3) tenglik ushbu

$$c_1 y^1(x) + c_2 y^2(x) + \dots + c_n y^n(x) = 0$$

ko'rinishni oladi. Bu esa $y^1(x), y^2(x), \dots, y^n(x)$ vektor-funksiyalarning chiziqli bog'langanligini ko'rsatadi. ■

4.10.1-izoh. Ixtiyoriy vektor-funksiyalar uchun 4.10.6-lemmaning tasdiqi bajarilmaydi. Jumladan, ushbu

$$y^1(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad y^2(x) = \begin{pmatrix} 0 \\ x \end{pmatrix}$$

vektor-funksiyalarning chiziqli bog'lanmagan, lekin

$$W(x) = W\{y^1(x), y^2(x)\} = \begin{vmatrix} 0 & 0 \\ 1 & x \end{vmatrix} = 0.$$

4.10.2-teorema. Aytaylik, (4.10.1) differensial tenglamalar sistemasining $y^1(x), y^2(x), \dots, y^n(x)$ yechimlaridan tuzilgan $W(x) = W\{y^1(x), y^2(x), \dots, y^n(x)\}$ – Vronskiy determinanti berilgan bo'lsin. U holda quyidagi

$$W(x) = W(x_0) \exp \left\{ \int_{x_0}^x spA(\tau) d\tau \right\}, \quad x \in [a, b] \quad (4.10.5)$$

Ostragradskiy-Liuwill formulasio'rinli. Bunda ushbu

$$spA(x) = a_{11}(x) + a_{22}(x) + \dots + a_{nn}(x) \quad (4.10.6)$$

munosabatga $A(x)$ matritsaning izi deyiladi.

Isbot. Avvalo $W(x)$ – Vronskiy determinanti uchun quyidagi

$$\frac{dW(x)}{dx} = (spA(x))W(x), \quad x \in [a, b] \quad (4.10.7)$$

differential tenglamani keltirib chiqaramiz.

Aytaylik, ushbu

$$y^j(x) = (y_1^j(x), y_2^j(x), \dots, y_n^j(x))^T, j = \overline{1, n}$$

ko‘rinishdagi vektor-funksiyalar (4.10.1) differensial tenglamalar sistemasining, ya’ni

$$\frac{dy^j(x)}{dx} = A(x)y^j(x)$$

yechimlaridan iborat bo‘lsin. U holda bu vektor-funksiyalarning $y_i^j(x)$ – koordinatasi

$$\frac{dy_i^j(x)}{dx} = \sum_{r=1}^n a_{ir}(x)y_r^j(x) \quad (4.10.8)$$

differential tenglamani qanoatlantiradi.

Endi $W(x)$ – Vronskiy determinantning hosilasini hisoblaymiz:

$$\frac{dW(x)}{dx} = \sum_{i,j=1}^n \frac{\partial W(x)}{\partial y_i^j(x)} \cdot \frac{dy_i^j(x)}{dx}. \quad (4.10.9)$$

Agar $y_i^r(x)$ elementning algebraik to‘ldiruvchisini $W_{ir}(x)$ – orqali belgilab, $W(x)$ – Vronskiy determinantini i – satr elementlari bo‘yicha yoysak, ushbu

$$W(x) = \sum_{r=1}^n y_i^r(x)W_{ir}(x) = \sum_{r=1}^{j-1} y_i^r(x)W_{ir}(x) + y_i^j(x)W_{ij}(x) + \sum_{r=j+1}^n y_i^r(x)W_{ir}(x)$$

yoyilma hosil bo‘ladi. Bundan

$$\frac{dW(x)}{dy_i^j(x)} = W_{ij}(x) \quad (4.10.10)$$

tenglikni olamiz. Endi (4.10.9) tenglikni (4.10.8) va (4.10.10) munosabatlardan foydalanib quyidagicha yozish mumkin:

$$\frac{dW(x)}{dx} = \sum_{i,j=1}^n W_{ij}(x) \sum_{r=1}^n a_{ir}(x)y_r^j(x) = \sum_{i,r=1}^n a_{ir}(x) \sum_{j=1}^n y_r^j(x)W_{ij}(x). \quad (4.10.11)$$

Quyidagi formula algebra kursidan ma’lum:

$$\sum_{j=1}^n y_r^j(x)W_{ij}(x) = W(x)\delta_{ri} = \begin{cases} W(x), & i = r, \\ 0, & i \neq r. \end{cases}$$

Bu yerda δ_{ri} – Kroneker simvoli. Shuning uchun yuqoridagi (4.10.11) differensial tenglamani

$$\frac{dW(x)}{dx} = \left\{ \sum_{i,r=1}^n a_{ir}(x)\delta_{ir} \right\} W(x) = \left\{ \sum_{i=1}^n a_{ii}(x)\delta_{ii} \right\} W(x) = W(x)spA(x)$$

ko‘rinishda yozish mumkin. Oxirgi differensial tenglamani integrallab, (4.10.5) Ostragradskiy-Liuvill formulasini hosil qilamiz. ■

4.10.2-natija. Agar A sonli matritsa bo‘lsa, u holda ushbu

$$\det e^A = e^{spA} \quad (4.10.12)$$

tenglik bajariladi.

Isbot. (4.10.5) Ostragradskiy-Liuivill formulasini ushbu

$$W(x) = \det e^{xA}$$

funksiya uchun $x_0 = 0, x = 1$ deb qo'llasak, (4.10.12) tenglik kelib chiqadi. ■

4.10.1-ta'rif. Bir jinsli (4.10.1) ko'rinishdagi chiziqli differensial tenglamalar sistemasining n ta chiziqli bog'lanmagan $y^1(x), y^2(x), \dots, y^n(x)$ yechimlariga, uning fundamental yechimlari sistemasi (F.Y.S) deyiladi. Bu yerda

$$y^j(x) = (y_1^j(x), y_2^j(x), \dots, y_n^j(x))^T, j = \overline{1, n}.$$

4.10.3-teorema. Uzluksiz koeffitsiyentli, (4.10.1) ko'rinishdagi bir jinsli differensial tenglamalar sistemasining fundamental yechimlari sistemasi (F.Y.S) mavjud.

Isbot. Faraz qilaylik, R^n fazoning e_1, e_2, \dots, e_n chiziqli bog'lanmagan vektorlari berilgan bo'lsin. U holda (4.10.1) differensial tenglamalar sistemasiga qo'yilgan ushbu

$$y^1(x_0) = e_1, y^2(x_0) = e_2, \dots, y^n(x_0) = e_n$$

Koshi masalasining yechimlarini $y^1(x), y^2(x), \dots, y^n(x)$ orqali belgilaymiz. Bu yechimlardan tuzilgan $W\{y^1(x), y^2(x), \dots, y^n(x)\} = W(x)$ Vronskiy determinantini hisoblaymiz:

$$W(x_0) = \det(e_1, e_2, \dots, e_n) \neq 0.$$

Chunki e_j – vektorlar chiziqli bog'lanmagan. Bundan esa $y^1(x), y^2(x), \dots, y^n(x)$ yechimlarning chiziqli bog'lanmaganligi kelib chiqadi.

■

4.10.3-teorema. Agar $y^1(x), y^2(x), \dots, y^n(x)$ vektor-funksiyalar (4.10.1) differensial tenglamalar sistemasining fundamental yechimlari sistemasini (F.Y.S) tashkil qilsa, u holda uning ixtiyoriy yechimi

$$y(x) = c_1 y^1(x) + c_2 y^2(x) + \dots + c_n y^n(x) \quad (4.10.13)$$

ko'rinishda ifodalanadi.

4.10.3-teoremaning isbotini bayon qilish o'rniga (4.10.13) formulaning boshqacha ko'rinishini keltiramiz.

4.10.3-ta'rif. Ustunlari fundamental yechimlar sistemasidan tuzilgan

$$Y(x) = \left\| y^1(x), y^2(x), \dots, y^n(x) \right\|$$

matritsa-funksiya (4.10.1) sistemaning fundamental matritsasi deyiladi. Fundamental matritsa o'z navbatida ushbu

$$\frac{dY(x)}{dx} = A(x)Y(x), \forall x \in [a, b], \det Y(x) \neq 0 \quad (4.10.14)$$

matritsaviy differensial tenglamani qanoatlantiradi. Bundan ko‘rinadiki, (4.10.1) sistemaning ixtiyoriy yechimi

$$y(x) = Y(x) \cdot C$$

ko‘rinishda yoziladi. Bu yerda C – ixtiyoriy sonli vektor.

4.10.4-teorema. Agar $\tilde{Y}(x)$ -(4.10.14) matritsaviy differensial tenglamaning fundamental matritsasi bo‘lsa, u holda uning umumiy yechimi

$$Y(x) = \tilde{Y}(x)C$$

ko‘rinishda bo‘ladi.

Isbot. (4.10.14) matritsaviy tenglamaning yechimini

$$Y(x) = \tilde{Y}(x)C(x)$$

ko‘rinishda izlaymiz. Ushbu

$$\frac{dY(x)}{dx} = \frac{d}{dx}(\tilde{Y}(x)C(x)) = \frac{d\tilde{Y}(x)}{dx}C(x) + \tilde{Y}(x)\frac{dC(x)}{dx}$$

tenglikdan foydalanib, (4.10.14) differensial tenglamadan

$$\frac{d\tilde{Y}(x)}{dx}C(x) + \tilde{Y}(x)\frac{dC(x)}{dx} = A(x)\tilde{Y}(x)C(x)$$

munosabatni hosil qilamiz. Bunda $\tilde{Y}(x)$ – matritsa (4.10.14) tenglamaning yechimi bo‘lgani uchun oxirgi tenglikdan

$$A(x)\tilde{Y}(x)C(x) + \tilde{Y}(x)\frac{dC(x)}{dx} = A(x)\tilde{Y}(x)C(x)$$

kelib chiqadi. Bundan

$$\tilde{Y}(x)\frac{dC(x)}{dx} = 0, \det \tilde{Y}(x) \neq 0,$$

ya’ni

$$\frac{dC(x)}{dx} = 0$$

differensial tenglamani topamiz. Oxirgi tenglamadan $C(x) \equiv C$ o‘zgarmas matritsa ekanligi kelib chiqadi. ■

Berilgan (4.10.1) differensial tenglamalar sistemasiga qo‘yilgan ushbu

$$y(x_0) = y^0$$

Koshi masalasining yechimi uchun

$$y(x) = Y(x) \cdot Y^{-1}(x_0) \cdot y^0$$

formulao‘rinli bo‘ladi. Bu yerda $Y(x)$ (4.10.14) matritsaviy differensial tenglamaning ushbu

$$Y(x_0) = E, E – birlik matritsa$$

boshlang‘ich shartni qanoatlantiruvchi yechimi.

Ushbu

$$K(x, x) = Y(x)Y^{-1}(x_0)$$

matritsaga (4.10.1) sistemaning matritsanti deyiladi.

11-§. Chiziqli bir jinsli bo‘lmagan differensial tenglamalar sistemasi

Quyidagi

$$\frac{dy}{dx} = A(x)y + f(x) \quad (4.11.1)$$

chiziqli bir jinsli bo‘lmagan differensial tenglamalar sistemasini qaraylik. Bu yerda $A(x)$, $x \in [a, b]$ oraliqda aniqlangan uzluksiz n -o‘lchamli kvadrat matritsa-funksiya, $f(x)$, $x \in [a, b]$ oraliqda aniqlangan uzluksiz vektor-funksiya, $y(x)$ – noma’lum vektor-funksiya.

4.11.1-lemma. Agar $y_1(x)$ va $y_2(x)$ vektor-funksiyalar mos ravishda ushbu

$$\begin{aligned} \frac{dy_1}{dx} &= A(x)y_1 + f_1(x), \\ \frac{dy_2}{dx} &= A(x)y_2 + f_2(x) \end{aligned}$$

tenglamalar sistemasining yechimlari bo‘lsa, u holda $y(x) = y_1(x) + y_2(x)$ vektor-funksiya quyidagi

$$\frac{dy}{dx} = A(x)y + f_1(x) + f_2(x)$$

differensial tenglamalar sistemasining yechimi bo‘ladi.

4.11.1-teorema. Aytaylik, $y^0(x)$ vektor-funksiya (4.11.1) sistemaning biror yechimi bo‘lib, $\Phi(x)$ ushbu

$$z'(x) = A(x)z(x) \quad (4.11.2)$$

bir jinsli tenglamalar sistemasining fundamental matritsasi bo‘lsin. U holda (4.11.1) differensial tenglamalar sistemasining barcha yechimlari

$$y(x) = \Phi(x)C + y^0(x) \quad (4.11.3)$$

ko‘rinishda ifodalanadi. Bunda C – ixtiyoriy sonli vektor.

Isbot. Berilgan (4.11.1) sistemada

$$y(x) = z(x) + y^0(x) \quad (4.11.4)$$

almashtirish bajaramiz:

$$\frac{dz(x)}{dx} + \frac{dy^0(x)}{dx} = A(x)[z(x) + y^0(x)] + f(x) = Az(x) + Ay^0(x) + f(x).$$

Bu yerda

$$\frac{dy^0(x)}{dx} = Ay^0(x) + f(x)$$

munosabatning bajarilishini e‘tiborga olsak, yuqoridagi tenglikdan

$$\frac{dz(x)}{dx} = A(x)z(x)$$

tenglama kelib chiqadi. (4.11.2) sistemaning umumiy yechimi

$$z(x) = \Phi(x)C$$

ko‘rinishda bo‘lgani uchun (4.11.4) almashtirishdan

$$y(x) = \Phi(x)C + y^0(x)$$

kelib chiqadi. Bu yerda C – ixtiyoriy sonli vektor. ■

4.11.2-teorema. Agar $\Phi(x)$ – (4.11.2) sistemaning fundamental matritsasi bo‘lsa, u holda (4.11.1) differensial tenglamalar sistemasining umumiy yechimi uchun

$$y(x) = \Phi(x)D + \Phi(x) \int_{x_0}^x \Phi^{-1}(\tau) f(\tau) d\tau \quad (4.11.5)$$

formula o‘rinli bo‘ladi. Bu yerda $x_0 \in [a, b]$, D – ixtiyoriy o‘zgarmas vektor.

Isbot. Berilgan (4.11.1) sistemaning umumiy yechimini

$$y(x) = \Phi(x) \cdot C(x) \quad (4.11.6)$$

ko‘rinishda izlaymiz. Bu yerda $C(x)$ – hozircha noma’lum vektor-funksiya. Avvalo (4.11.6) tenglikni differensiallab,

$$\frac{dy}{dx} = \frac{d\Phi(x)}{dx} C(x) + \Phi(x) \frac{dC(x)}{dx}$$

munosabatni topamiz. So‘ngra bu tenglikni (4.11.1) sistemaga qo‘ysak,

$$\frac{d\Phi(x)}{dx} C(x) + \Phi(x) \frac{dC(x)}{dx} = A(x)\Phi(x)C(x) + f(x) \quad (4.11.7)$$

hosil bo‘ladi. Agar ushbu

$$\frac{d\Phi(x)}{dx} = A(x)\Phi(x)$$

tenglikning o‘rinli ekanligini inobatga olsak, (4.11.7) munosabatdan

$$A(x)\Phi(x)C(x) + \Phi(x) \frac{dC(x)}{dx} = A(x)\Phi(x)C(x) + f(x),$$

ya’ni

$$\Phi(x) \frac{dC(x)}{dx} = f(x)$$

tenglama kelib chiqadi. Bu tenglamani ushbu

$$\frac{dC(x)}{dx} = \Phi^{-1}(x) f(x) \quad (4.11.8)$$

ko‘rinishda yozib, so‘ngra integrallab,

$$C(x) = \int_{x_0}^x \Phi^{-1}(\tau) f(\tau) d\tau + D, \quad (4.11.9)$$

$$C(x_0) \equiv D$$

topamiz. (4.11.9) tenglikni (4.11.6) ga qo‘yib, ushbu

$$y(x) = \Phi(x)D + \Phi(x) \int_{x_0}^x \Phi^{-1}(\tau) f(\tau) d\tau$$

formulani hosil qilamiz. ■

4.11.1-natija. (4.11.1) differensial tenglamalar sistemasiga qo'yilgan

$$y(x_0) = y^0$$

Koshi masalasining yechimi uchun

$$y(x) = \Phi(x)\Phi^{-1}(x_0)y^0 + \int_{x_0}^x \Phi(x)\Phi^{-1}(\tau)f(\tau)d\tau \quad (4.11.10)$$

formula o'rinli bo'ladi. Bu yerda $x_0 \in [a, b]$.

4.11.1-izoh. 1) Quyidagi ko'rinishdagi vektor-funksiya

$$y_0(x) = \Phi(x) \int_{x_0}^x \Phi^{-1}(\tau) f(\tau) d\tau$$

(4.11.1) differensial tenglamalar sistemasining

$$y_0(x_0) = 0$$

boshlang'ich shartni qanoatlantiruvchi yechimidan iborat bo'ladi.

2) Ushbu

$$K(x, \tau) = \Phi(x)\Phi^{-1}(\tau)$$

matritsa-funksiya (4.11.1) sistemaning matritsantidan iborat bo'ladi, ya'ni

$$\frac{dK(x, \tau)}{dx} = A(x)K(x, \tau), \quad K(\tau, \tau) = E - \text{birlik matritsa.}$$

3) Agar (4.11.1) sistemada $A(x) = (a_{ij}), a_{ij} = \text{const}$ o'zgarmas matritsa bo'lsa, u holda uning umumiy yechimi uchun

$$y(x) = e^{xA} \cdot d + \int_{x_0}^x e^{(x-\tau)A} f(\tau) d\tau$$

formula o'rinli bo'ladi. Bu yerda $\Phi(x) = e^{xA}$ (4.11.2) sistemaning fundamental matritsasi bo'lib, d – ixtiyoriy sonli vektor.

4.11.1-misol. Ushbu

$$\frac{dy_1}{dx} = 4y_1 - 2y_2,$$

$$\frac{dy_2}{dx} = 8y_1 - 4y_2 + \sqrt{x}, \quad x \geq 0$$

bir jinsli bo'lmagan differensial tenglamalar sistemasini Lagranj usulidan foydalanib yeching.

Yechish. Avvalo, ushbu

$$\frac{dy_1}{dx} = 4y_1 - 2y_2,$$

$$\frac{dy_2}{dx} = 8y_1 - 4y_2$$

bir jinsli differensial tenglamalar sistemasining koeffitsiyentlaridan

$$A = \begin{vmatrix} 4 & -2 \\ 8 & -4 \end{vmatrix}$$

matritsa tuzib olamiz. Soʻngra bu matritsaning xos qiymatlarini va xos vektorlarini

topamiz. Buning uchun

$$Ah = \lambda h, 0 \neq h = (x_1, x_2)^T$$

tenglamani qaraymiz. Bu tenglamani koordinatalarda yozib

$$\begin{cases} 4x_1 - 2x_2 = \lambda x_1, \\ 8x_1 - 4x_2 = \lambda x_2, \end{cases}$$

yaʼni

$$\begin{cases} (4 - \lambda)x_1 - 2x_2 = 0, \\ 8x_1 + (-4 - \lambda)x_2 = 0 \end{cases}$$

bir jinsli algebraik tenglamalar sistemasini hosil qilamiz. Bundan

$$\begin{vmatrix} 4 - \lambda & -2 \\ 8 & -4 - \lambda \end{vmatrix} = 0, (4 - \lambda)(-4 - \lambda) + 16 = 0, \\ -(4 - \lambda)(4 + \lambda) + 16 = 0, -(16 - \lambda^2) + 16 = 0, \\ -16 + \lambda^2 + 16 = 0, \lambda^2 = 0, \lambda_1 = \lambda_2 = 0$$

karrali xos qiymatlarni topib olamiz.

$\lambda_1 = 0$ xos qiymatga mos keluvchi xos vektorni ushbu

$$\begin{cases} 4x_1 - 2x_2 = 0, \\ 8x_1 - 4x_2 = 0 \end{cases}$$

sistemadan aniqlaymiz: $x_1 = 1, x_2 = 2, h_1 = (1, 2)^T$.

Endi h_1 xos vektorga yopishgan h_2 -vektorni aniqlash uchun

$$(A - \lambda_2)h_1 = h_2, \lambda_2 = 0$$

tenglamadan foydalanamiz: $h_2 = \left(\frac{1}{4}, 0\right)^T$.

Bu maʼlumotlardan foydalanib, bir jinsli differensial tenglamalar sistemasining umumiy yechimini topamiz:

$$y(x) = c_1 h_1 + c_2 (x h_1 + h_2) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \left(x \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1/4 \\ 0 \end{pmatrix} \right).$$

Endi o'zgarmaning variatsiyalash usulidan foydalanib, berilgan bir jinsli bo'lmagan differensial tenglamalar sistemasining yechimini

$$y(x) = c_1(x) \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2(x) \left(x \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1/4 \\ 0 \end{pmatrix} \right)$$

ko'rinishda izlaymiz. Bu yerda $c_1(x)$, $c_2(x)$ – hozircha noma'lum funksiyalar:

$$\begin{aligned} y'(x) &= \begin{pmatrix} y_1'(x) \\ y_2'(x) \end{pmatrix} = \begin{pmatrix} c_1'(x) \\ 2c_1'(x) \end{pmatrix} + \begin{pmatrix} c_2(x)(x+1/4) \\ 2c_2(x)x \end{pmatrix}' = \begin{pmatrix} c_1'(x) \\ 2c_2'(x) \end{pmatrix} + \\ &+ \begin{pmatrix} c_2(x) + c_2'(x)(x+1/4) \\ 2c_2(x) + 2c_2'(x)x \end{pmatrix} = \begin{pmatrix} c_1'(x) + c_2(x) + c_2'(x)(x+1/4) \\ 2c_1'(x) + 2c_2(x) + 2c_2'(x)x \end{pmatrix}, \\ y_1'(x) &= c_1'(x) + c_2'(x)(x+1/4) + c_2(x), \\ y_2'(x) &= 2c_1'(x) + 2c_2'(x)x + 2c_2(x). \end{aligned}$$

Topilgan $y_1'(x)$, $y_2'(x)$ hosilalarning ifodalarini berilgan bir jinsli bo'lmagan differensial tenglamalar sistemasiga qo'yamiz:

$$\begin{cases} c_1'(x) + c_2'(x)(x+1/4) + c_2(x) = 4[c_1(x) + (x+1/4)c_2(x)] - 2[2c_1(x) + 2xc_2(x)], \\ 2c_1'(x) + 2c_2'(x)x + 2c_2(x) = 8[c_1(x) + (x+1/4)c_2(x)] - 4[2c_1(x) + 2xc_2(x)] + \sqrt{x} \end{cases}$$

$$\begin{cases} c_1' + c_2'(x+1/4) + c_2 = 4c_1 + 4c_2x + c_2 - 4c_1 - 4xc_2, \\ 2c_1' + 2c_2'x + 2c_2 = 8c_1 + 8xc_2 + 2c_2 - 8c_1 - 8xc_2 + \sqrt{x}, \end{cases}$$

$$\begin{cases} c_1' + (x+1/4)c_2' = 0, \\ 2c_1' + 2xc_2' = \sqrt{x}. \end{cases}$$

Bundan $c_1'(x) = (2x+1/2)\sqrt{x}$, $c_2'(x) = -2\sqrt{x}$ oddiy differensial tenglamalarni hosil qilamiz. Ularni integrallab, $c_1(x)$ va $c_2(x)$ larni topamiz:

$$c_1(x) = \left(\frac{1}{3}x + \frac{4}{5}x^2 \right) \sqrt{x} + A,$$

$$c_2(x) = -\frac{4}{3}x\sqrt{x} + B,$$

bu yerda A, B – ixtiyoriy haqiqiy o'zgarmaning sonlar. Nihoyat, ushbu

$$y(x) = Ah_1 + B(xh_1 + h_2) + \begin{pmatrix} -\frac{8}{15}x^2\sqrt{x} \\ -\frac{16}{15}x^2\sqrt{x} + \frac{2}{3}x\sqrt{x} \end{pmatrix}$$

yechimni topamiz.

V BOB. TURG‘UNLIK NAZARIYASI

1-§. Turg‘unlik tushunchasi

Aytaylik, ushbu

$$\frac{dx(t)}{dt} = f(t, x), \quad (5.1.1)$$

$$x(t_0) = x_0 \quad (5.1.2)$$

Koshi masalasining $x = \varphi(t)$ yechimi mavjud bo‘lib, ixtiyoriy $t \in [t_0, \infty)$ to‘plamda aniqlangan bo‘lsin. Bu yerda

$$x(t) = (x_1(t), x_2(t), x_3(t), \dots, x_n(t))^T,$$

$$f(t, x) = (f_1(t, x), \dots, f_n(t, x))^T,$$

$$\varphi(t) = (\varphi_1(t), \varphi_2(t), \varphi_3(t), \dots, \varphi_n(t))^T.$$

5.1.1-ta’rif. Agar ixtiyoriy $\varepsilon > 0$ soni uchun, shunday $\delta > 0$ soni topilib,

$$|\tilde{x}_0 - x_0| < \delta \quad (5.1.3)$$

tengsizlikni qanoatlantiruvchi har bir \tilde{x}_0 uchun quyidagi

$$\frac{d\tilde{x}(t)}{dt} = f(t, \tilde{x}) \quad (5.1.1')$$

$$\tilde{x}(t_0) = \tilde{x}_0 \quad (5.1.2')$$

Koshi masalasining $\tilde{x}(t)$, $t \in [t_0, \infty)$ yechimi mavjud bo‘lib, ushbu

$$|\tilde{x}(t) - \varphi(t)| < \varepsilon, \quad \forall t \in [t_0, \infty) \quad (5.1.4)$$

tengsizlikni qanoatlantirsa, u holda $x = \varphi(t)$ yechim Lyapunov ma’nosida turg‘un deyiladi.

5.1.2-ta’rif. Agar $x = \varphi(t)$, $t \geq t_0$ yechim

1) Lyapunov ma’nosida turg‘un;

2) Ushbu $\lim_{t \rightarrow \infty} |\tilde{x}(t) - \varphi(t)| = 0$ munosabat o‘rinli

bo‘lsa, unga asimptotik turg‘un yechim deyiladi.

Berilgan (5.1.1) differensial tenglamalar sistemasi $x = \varphi(t)$ yechimining turg‘unligini tekshirish masalasi, uning nol, ya’ni $x(t) \equiv 0$ yechimining turg‘unligini tekshirish masalasiga keltirish mumkin. Buning uchun

$$x(t) = \varphi(t) + y(t), \quad t \in [t_0, \infty) \quad (5.1.6)$$

almashtirishdan foydalanamiz. Bu almashtirish natijasida (5.1.1) differensial tenglama

$$\dot{\varphi}(t) + \dot{y}(t) = f(t, \varphi(t) + y(t)) \quad (5.1.7)$$

ko‘rinishni oladi. Bunda ushbu

$$\dot{\varphi}(t) = f(t, \varphi(t))$$

munosabatning bajarilishini inobatga olsak, (5.1.7) tenglik quyidagi

$$\dot{y}(t) = f(t, \varphi(t) + y(t)) - f(t, \varphi(t)), t \geq t_0 \quad (5.1.8)$$

ko‘rinishga keladi. Berilgan (5.1.1) differensial tenglamaning $x = \varphi(t)$ yechimi (5.1.6) almashtirish natijasida (5.1.8) tenglamaning $y(t) \equiv 0$ nol yechimiga o‘tadi. Endi, (5.1.8) tenglamani

$$\dot{y}(t) = F(t, y), F(t, y) = f(t, \varphi(t) + y(t)) - f(t, \varphi(t)) \quad (5.1.9)$$

ko‘rinishda yozamiz. Bu holda $y(t) = 0$ yechimga, ya‘ni $(0, \dots, 0)$ nuqtaga (5.1.9) differensial tenglamalar sistemasining muvozanat nuqtasi deyiladi. Chunki

$$F(t, 0) = f(t, \varphi) - f(t, \varphi) = 0.$$

Turg‘unlik tushunchasi (5.1.9) differensial tenglamalar sistemasining muvozanat nuqtasiga, ya‘ni $y(t) \equiv 0$ yechimga nisbatan quyidagicha talqin qilinadi.

5.1.3-ta‘rif. Agar $\forall \varepsilon > 0$ soni uchun shunday $\exists \delta > 0$ soni topilib,

$$|y_0| < \delta$$

tengsizlikni qanoatlantiruvchi har bir y_0 uchun (5.1.9) sistemaning $y_0(t_0) = y_0$ boshlang‘ich shartni qanoatlantiruvchi $y(t)$, $t \in [t_0, \infty)$ yechimi

$$|y(t)| < \varepsilon, t \in [t_0, \infty)$$

bahoni qanoatlantirsa, u holda $y(t) \equiv 0$ yechim, ya‘ni $(0, \dots, 0)$ muvozanat nuqta Lyapunov ma‘nosida turg‘un deyiladi.

5.1.4-ta‘rif. Agar $y(t) \equiv 0$ yechim (muvozanat nuqta) quyidagi:

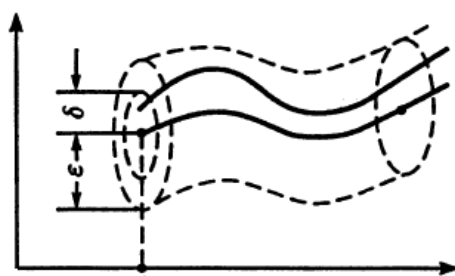
- 1). Lyapunov ma‘nosida turg‘un;
- 2). $\lim_{t \rightarrow \infty} |y(t)| = 0$

shartlarni qanoatlantirsa, muvozanat nuqta asimptotik turg‘un deyiladi.

Yuqoridagi ta‘riflarda $|\cdot|$ belgi vektor funksiyaning normasini anglatadi, ya‘ni $|x(t)| = \sqrt{x_1^2(t) + \dots + x_n^2(t)}$.

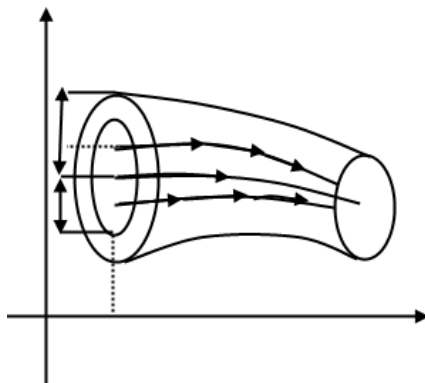
Geometrik nuqtaiy nazardan $\varphi(t)$ yechimning turg‘unligini quyidagicha tasvirlash mumkin:

- 1) $\varphi(t)$ yechim turg‘un, $y = (y_1, y_2, \dots, y_n)^T$:



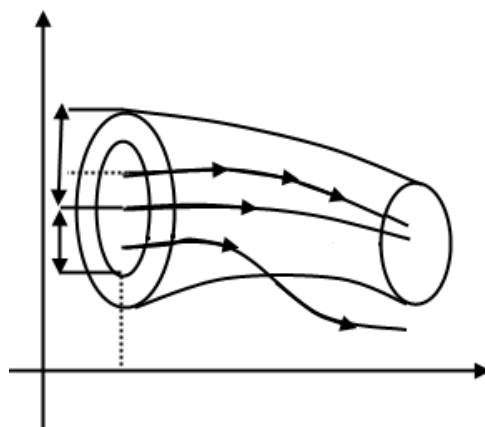
1-chizma

2) $\varphi(t)$ yechim asimptotik turg'un:



2-chizma

3) $\varphi(t)$ yechim turg'un emas (noturg'un)



3-chizma

5.1.1-misol. Ushbu

$$\frac{dy}{dt} = -y^2$$

differensial tenglamaning $y(t) \equiv 0$ yechimini turg'unlikka tekshiring.

Yechish. Berilgan differensial tenglamani

$$-\frac{dy}{y^2} = dt$$

ko'rinishda yozib olamiz va uning umumiy yechimini topamiz:

$$y(t) = \frac{1}{t+c}, \quad \forall c = const.$$

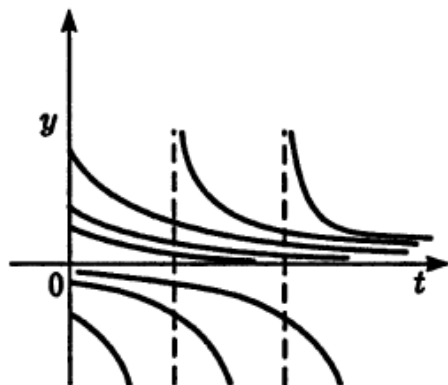
Bundan tashqari $y(t) = 0$ ham berilgan differensial tenglamaning yechimidan iborat bo'ladi. Endi $y(0) = y_0$ boshlang'ich shartga mos keluvchi yechimni aniqlaymiz:

$$y(0) = y_0 \Rightarrow c = \frac{1}{y_0}; \quad y(t) = \frac{1}{t + \frac{1}{y_0}} = \frac{y_0}{ty_0 + 1}.$$

Berilgan differensial tenglamaning $y(t) = 0$ yechimini turg'unlikka tekshirishda, ushbu $|y(0)| = |y_0| < \delta$ tengsizlikning bajarilishidan

$|y(t)| < \varepsilon, 0 \leq t < \infty$ bahoning kelib chiqishini ko'rsatish lozim. Ammo, qaralayotgan misolda $y_0 < 0$ bo'lganda $ty_0 + 1 = 0, t = -\frac{1}{y_0} > 0$ bo'ladi.

Shuning uchun $t \rightarrow -\frac{1}{y_0} - 0$ da $y(t) \rightarrow -\infty$ intiladi. Bu holda $y(t) \equiv 0$ yechim turg'un bo'lmaydi.



4-chizma

5.1.2-misol. Ushbu

$$\frac{dy}{dt} = 1 + t - y, \quad y(0) = 0$$

Koshi masalasi yechimini turg'unlikka tekshiring.

Yechish. Avvalo berilgan differensial tenglamani

$$\frac{dy}{dt} + y = 1 + t$$

ko'rinishda yozib olamiz va uning bir jinsli

$$\frac{dy}{dt} + y = 0$$

qismining umumiy yechimini topamiz:

$$y(t) = ce^{-t}, \quad c = \text{const}.$$

So'ngra, berilgan bir jinsli bo'lmagan differensial tenglamaning xususiy yechimini

$$y_0(t) = At + B$$

ko'rinishda izlab, A va B noma'lumlarning qiymatlarini aniqlaymiz:

$$y_0'(t) = A, \quad A + At + B = 1 + t; \quad A = 1, \quad B = 0.$$

Demak, xususiy yechim ushbu

$$y_0(t) = t$$

ko'rinishda bo'lar ekan. Shuning uchun berilgan bir jinsli bo'lmagan differensial tenglamaning umumiy yechimi

$$y(t) = ce^{-t} + y_0(t) = ce^{-t} + t$$

ko‘rinishida bo‘ladi. Endi $y(0) = y_0$ boshlang‘ich shartdan $c = y_0$ ekanligini topamiz. Natijada ushbu $y(t) = y_0 e^{-t} + t$ yechimni aniqlaymiz. Bu yerda $y_0 = 0$ bo‘lsa, u holda

$$y(t) = \varphi(t) = t$$

funksiya berilgan Koshi masalasining yagona yechimidan iborat bo‘ladi. Bu $\varphi(t) = t$ yechimni turg‘unlikka tekshiramiz. Buning uchun quyidagi ayirmani baholaymiz:

$$|y(t) - \varphi(t)| = |y_0 e^{-t} + t - t| = |y_0| e^{-t} \leq |y_0| < \varepsilon, \forall t \geq 0.$$

Shunday qilib, agar $\delta = \varepsilon$ desak, u holda $|y_0| < \delta$ tengsizlikning bajarilishidan $|y(t) - \varphi(t)| = |y(t) - t| < \varepsilon, \forall t \geq 0$ bahoning o‘rinli bo‘lishi kelib chiqadi. Bu esa $\varphi(t) = t$ yechimning Lyapunov ma‘nosida turg‘un ekanligini ko‘rsatadi. Bundan tashqari $\varphi(t) = t$ yechim asimptotik turg‘un yechim ham bo‘ladi. Chunki

$$\lim_{t \rightarrow \infty} |y(t) - \varphi(t)| = \lim_{t \rightarrow \infty} |y_0| e^{-t} = 0, \forall t \geq 0.$$

5.1.3-misol. Ushbu

$$\frac{dy}{dt} = -a^2 y, y(t_0) = y_0, a \neq 0$$

Koshi masalasining yechimini turg‘unlikka tekshiring.

Yechish. Berilgan differensial tenglamaning umumiy yechimi $y(t) = c e^{-a^2 t}$, $\forall c = const$ ko‘rinishda bo‘ladi. $y(t_0) = y_0$ boshlang‘ich shartdan $c = y_0 e^{a^2 t_0}$ kelib chiqadi. Bundan foydalanib, berilgan Koshi masalasining

$$y(t) = y_0 e^{-a^2(t-t_0)}$$

yechimini topamiz. Endi $|y_0 - \tilde{y}_0| < \delta$, $\delta > 0$ tengsizlikni qanoatlantiruvchi \tilde{y}_0 sonini tanlaymiz va $y(t_0) = \tilde{y}_0$ boshlang‘ich shartga mos keluvchi Koshi masalasining $\tilde{y}(t) = \tilde{y}_0 e^{-a^2(t-t_0)}$ yechimini olamiz. Quyidagi ayirmani bajaramiz:

$$|\tilde{y}(t) - y(t)| = |\tilde{y}_0 - y_0| e^{-a^2(t-t_0)} = |\tilde{y}_0 - y_0| e^{a^2 t_0} \cdot e^{-a^2 t} \leq |\tilde{y}_0 - y_0| e^{a^2 t_0} < \varepsilon, \forall t \geq t_0.$$

Agar $\forall \varepsilon > 0$ soni uchun $\delta > 0$ sonini $\delta = \varepsilon e^{-a^2 t_0}$ deb tanlasak, u holda $|\tilde{y}_0 - y_0| < \delta$ tengsizlikning bajarilishidan $|\tilde{y}(t) - y(t)| < \varepsilon, \forall t \geq t_0$ bahoning o‘rinli bo‘lishi kelib chiqadi. Qaralayotgan misolda $y(t) = y_0 \exp\{-a^2(t-t_0)\}, t \geq t_0$ yechim Lyapunov ma‘nosida turg‘un bo‘ladi.

Bundan tashqari ushbu $\lim_{t \rightarrow \infty} |\tilde{y}(t) - y(t)| = \lim_{t \rightarrow \infty} |\tilde{y}_0 - y_0| \exp\{-a^2(t-t_0)\} = 0, t \geq t_0$ munosabatning bajarilishi-dan, berilgan Koshi masalasining $y(t) = y_0 \exp\{-a^2(t-t_0)\}, t \geq t_0$ yechimi asimptotik turg‘un bo‘lishi kelib chiqadi.

5.1.4-misol. Ushbu

$$\frac{dy}{dt} = \frac{a}{t} y, \quad y(1) = 0$$

Koshi masalasining yechimini turg'unlikka tekshiring.

Yechish. Ko'rinib turibdiki, ushbu $\tilde{y}(1) = y_0$ Koshi masalasining yechimi $\tilde{y}(t) = t^a y_0$ ko'rinishida bo'ladi. Berilgan $y(1) = 0$ boshlang'ich shartni qanoatlantiruvchi yechim esa $y(t) \equiv 0$ dan iborat bo'ladi.

Quyidagi ayirmani hisoblaylik:

$$|\tilde{y}(t) - y(t)| = |t^a y_0 - 0| = |t|^a \cdot |y_0|.$$

1-hol. Agar $a = 0$ bo'lsa, u holda

$$|\tilde{y}(t) - y(t)| = |y_0|$$

munosabat o'rinli bo'ladi. Bundan ko'rinadiki, $\forall \varepsilon > 0$ soni uchun $\delta > 0$ sonini $\delta = \varepsilon$ deb olsak u holda $|y_0| < \delta$ tengsizlikning bajarilishidan $|\tilde{y}(t) - y(t)| = |\tilde{y}(t)| = |y_0| < \varepsilon, \forall t \geq 1$, bahoning o'rinli bo'lishi kelib chiqadi. Bu esa $y(t) \equiv 0$ yechimning Lyapunov ma'nosida turg'unligini ko'rsatadi.

2-hol. Agar $a < 0$ bo'lsa, u holda

$$\lim_{t \rightarrow \infty} |\tilde{y}(t) - y(t)| = \lim_{t \rightarrow \infty} |t|^a |y_0| = 0$$

munosabatning bajarilishidan $y(t) \equiv 0$ yechimning asimptotik turg'unligi kelib chiqadi.

3-hol. Agar $a > 0$ bo'lsa, u holda $|\tilde{y}(t) - y(t)| = |t|^a \cdot |y_0| \rightarrow +\infty, t \rightarrow \infty$ bo'lgani uchun $y(t) \equiv 0$ yechim noturg'un bo'ladi. Demak, $a = 0$ holida $y(t) \equiv 0$ yechim turg'un, $a < 0$ holida esa asimptotik turg'un va $a > 0$ holida turg'unmas (noturg'un) bo'ladi.

5.1.5-misol. Ushbu

$$\begin{cases} \frac{dx_1}{dt} = -2x_2, \\ \frac{dx_2}{dt} = 2x_1 \end{cases}$$

differensial tenglamalar sistemasining muvozanat nuqtalarini toping va uni turg'unlikka tekshiring.

Yechish. Quyidagi tenglamalar sistemasidan

$$\begin{cases} -2x_2 = 0, \\ 2x_1 = 0 \end{cases}$$

$(x_1, x_2) = (0, 0)$ nuqtaning muvozanat nuqta ekanligi kelib chiqadi.

Berilgan differensial tenglamalar sistemasining umumiy yechimi

$$x_1(t) = c \cos(2t - \varphi), \quad x_2(t) = c \sin(2t - \varphi)$$

ko'rinishda bo'lishini ko'rsatish qiyinchilik tug'dirmaydi. Bunda $t = 0$ deb,

$$x(0) = (x_1(0), x_2(0)) = (c \cos \varphi, -c \sin \varphi)$$

ekanligini topamiz. Endi quyidagi ayirmalarni baholaymiz:

$$|x(0) - 0| = |x(0)| = \sqrt{x_1^2(0) + x_2^2(0)} = \sqrt{c^2 \cos^2 \varphi + c^2 \sin^2 \varphi} = |c|,$$

$$|x(t) - 0| = |x(t)| = \sqrt{x_1^2(t) + x_2^2(t)} = \sqrt{c^2 \cos^2(2t - \varphi) + c^2 \sin^2(2t - \varphi)} = |c|.$$

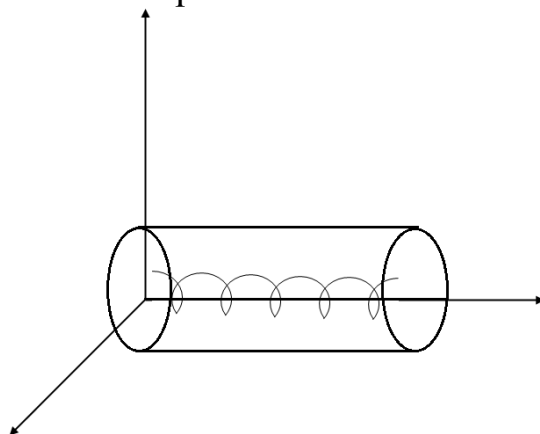
Ixtiyoriy $\forall \varepsilon > 0$ soni uchun $\exists \delta(\varepsilon) > 0$ sonini $\delta = \varepsilon$ deb tanlasak, u holda

$$|x(0) - 0| = |c| < \delta$$

tengsizlikning bajarilishidan

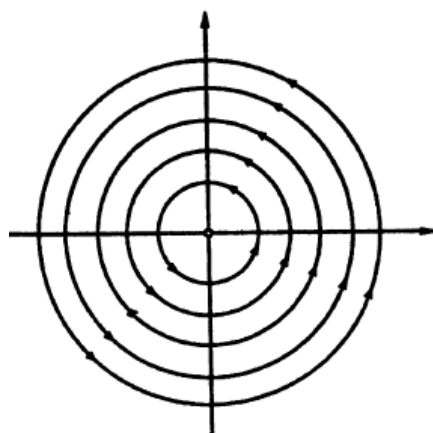
$$|x(t) - 0| = |x(t)| < \varepsilon, \quad \forall t \geq 0$$

bahoning o'rinli bo'lishi kelib chiqadi. Demak, $(0,0)$ muvozanat nuqta Lyapunov ma'nosida turg'un bo'lar ekan. Ammo, $(0,0)$ muvozanat nuqta asimptotik turg'un bo'lmaydi. Chunki $\lim_{t \rightarrow \infty} |x(t)| = |c| \neq 0$. Qaralayotgan misolda integral chiziqlar spirallardan iborat bo'ladi. Ularning Ox_1x_2 tekislikdagi proyeksiyasi markazi $(0,0)$ nuqtada bo'lgan aylanalardan iborat bo'ladi. Bu holda muvozanat nuqta markaz deb ataladi.



5-chizma

Integral chiziqlar spirallardan iborat bo'ladi.



6-chizma

Integral chiziqlarning Ox_1x_2 tekislikdagi proyeksiyasi (trayektoriyasi) markazi $(0,0)$ nuqtada bo'lgan aylanalardan iborat bo'ladi.

5.1.6-misol. Ushbu

$$\begin{cases} \frac{dx_1}{dt} = -3x_1, \\ \frac{dx_2}{dt} = -x_2 \end{cases}$$

differensial tenglamalar sistemasining muvozanat nuqtasini turg'unlikka tekshiring.

Yechish. Ko'rinib turibdiki, $(0,0)$ nuqta berilgan differensial tenglamalar sistemasi uchun muvozanat nuqta bo'ladi. Bundan tashqari differensial tenglamalar sistemasining umumiy yechimi

$$x_1(t) = x_1(0)e^{-3t}, \quad x_2(t) = x_2(0)e^{-t}$$

formula bilan aniqlanadi. Ushbu $|x(0)| = \sqrt{x_1^2(0) + x_2^2(0)}$, $|x(t)| = \sqrt{x_1^2(0)e^{-6t} + x_2^2(0)e^{-2t}} < e^{-t}|x(0)| \leq |x(0)|$ munosabatlarga asosan, $\forall \varepsilon > 0$ soni uchun $\delta > 0$ sonini $\delta = \varepsilon$ deb tanlasak $|x(0)| < \delta$ tengsizlikning bajarilishidan $|x(t)| \leq |x(0)| < \varepsilon, \forall t \geq 0$ bahoning o'rinli bo'lishi kelib chiqadi. Bu esa $(0,0)$ muvozanat nuqtaning Lyapunov ma'nosida turg'un ekanligini ko'rsatadi. Bundan tashqari ushbu

$$\lim_{t \rightarrow \infty} |x(t)| = \lim_{t \rightarrow \infty} \left| \sqrt{x_1^2(t) + x_2^2(t)} - 0 \right| = 0$$

munosabatning bajarilishidan $(0,0)$ muvozanat nuqtaning asimptotik turg'unligi ham kelib chiqadi.

5.1.7-misol. Quyidagi differensial tenglamalar sistemasining muvozanat nuqtasini turg'unlikka tekshiring:

$$\begin{cases} \frac{dx_1}{dt} = x_2, \\ \frac{dx_2}{dt} = -x_1. \end{cases}$$

Yechish. Avvalo berilgan differensial tenglamalar sistemasining umumiy yechimini topamiz:

$$x_1(t) = c_1 \cos t + c_2 \sin t, \quad x_2(t) = -c_1 \sin t + c_2 \cos t, \quad t \geq 0.$$

Endi $x(0) = (x_1(0), x_2(0))$ boshlang'ich shartni qanotlantiruvchi yechimni topamiz:

$$x_1(t) = x_1(0) \cos t + x_2(0) \sin t, \quad x_2(t) = -x_1(0) \sin t + x_2(0) \cos t, \quad t \geq 0.$$

Ixtiyoriy $\varepsilon > 0$ soni uchun $\exists \delta = \delta(\varepsilon) > 0$ sonini $\delta = \varepsilon$ deb tanlaymiz.

Bu holda

$$|x(0)| = |(x_1(0), x_2(0))| = \sqrt{x_1^2(0) + x_2^2(0)} < \delta$$

tengsizligi bajarilganda

$$|x(t)| = \sqrt{x_1^2(t) + x_2^2(t)} = \sqrt{(x_1(0)\cos t + x_2(0)\sin t)^2 + (-x_1(0)\sin t + x_2(0)\cos t)^2} = \\ = \sqrt{x_1^2(0) + x_2^2(0)} < \varepsilon, \quad \forall t \geq 0$$

bahoning o'rinli bo'lishi kelib chiqadi. Bundan $(0,0)$ muvozanat nuqtaning Lyapunov ma'nosida turg'unligi kelib chiqadi.

Quyidagi

$$\lim_{t \rightarrow \infty} |x(t)| = \sqrt{x_1^2(0) + x_2^2(0)} \neq 0$$

munosabatdan esa $(0,0)$ muvozanat nuqtaning asimptotik turg'un emasligi kelib chiqadi

5.1.8-misol. Quyidagi differensial tenglamalar sistemasining muvozanat nuqtasini turg'unlikka tekshiring:

$$\begin{cases} \frac{dx_1}{dt} = x_1, \\ \frac{dx_2}{dt} = -x_2. \end{cases}$$

Yechish. Ko'rinib turibdiki, $(0,0)$ nuqta berilgan differensial tenglamalar sistemasi uchun muvozanat nuqta bo'ladi. Berilgan differensial tenglamalar sistemasining umumiy yechimi

$$x_1(t) = x_1(0)e^t, \quad x_2(t) = x_2(0)e^{-t}$$

ko'rinishda bo'ladi. Bunda

$$x(0) = (x_1(0), x_2(0)), \quad x(t) = (x_1(t), x_2(t)).$$

Agar $x_1(0) > 0, (x_1(0) < 0)$ bo'lsa, u holda $x_1(t) \rightarrow \infty, t \rightarrow \infty,$
 $(x_1(t) \rightarrow -\infty, t \rightarrow \infty).$

Bundan $(0,0)$ muvozanat nuqtaning turg'unmasligi kelib chiqadi.

2-§. Chiziqli bir jinsli differensial tenglamalar sistemasi yechimining turg'unligi

Ushbu

$$\frac{dx(t)}{dt} = A(t)x(t) \quad (5.2.1)$$

differensial tenglamalar sistemasini qaraylik. Bu yerda $A(t) = \|a_{ij}(t)\|, i, j = \overline{1, n}$ kvadrat matritsa funksiya $t \geq t_0$ tengsizlikni qanoatlantiruvchi barcha t larda uzluksiz va chegaralangan bo'lsin.

Aytaylik, $x^j(t), j = \overline{1, n}$ vektor-funksiyalar (5.2.1) sistemaning ushbu

$$x^1(t_0) = e_1, x^2(t_0) = e_2, \dots, x^n(t_0) = e_n \quad (5.2.2)$$

boshlang'ich shartlarni qanoatlantiruvchi yechimlaridan iborat bo'lsin. Bunda

$$e_1 = (1, 0, 0, \dots, 0), e_2 = (0, 1, 0, \dots, 0), \dots, e_n = (0, 0, \dots, 0, 1). \quad (5.2.3)$$

Berilgan (5.2.1)-(5.2.2) Koshi masalasining $x^j(t), j = \overline{1, n}$ yechimlaridan tuzilgan $W(t)$ -Vronskiy determinantining $t = t_0$ nuqtadagi qiymati uchun

$$W(t_0) = \begin{vmatrix} x_1^1(t_0) & x_1^2(t_0) & \dots & x_1^n(t_0) \\ x_2^1(t_0) & x_2^2(t_0) & \dots & x_2^n(t_0) \\ \dots & \dots & \dots & \dots \\ x_n^1(t_0) & x_n^2(t_0) & \dots & x_n^n(t_0) \end{vmatrix} = \det E = 1 \neq 0$$

munosabat o'rinli. Shuning uchun $x^j(t), j = \overline{1, n}$ vektor-funksiyalar (5.2.1) differensial tenglamalar sistemasining chiziqli erkli yechimlari bo'ladi. Ushbu

$$X(t, t_0) = \|x_i^j(t)\|; i, j = \overline{1, n}$$

matritsa funksiya (5.2.1) differensial tenglamalar sistemasining fundamental matritsasi yoki matritsanti bo'lib, $X(t_0, t_0) = E$ boshlang'ich shartni va quyidagi

$$\frac{dX}{dt} = A(t)X, \det X(t, t_0) \neq 0$$

matrisaviy differensial tenglamani qanoatlantiradi. Bu yerda E -birlik matritsa. Bundan foydalanib (5.2.1) sistemaning $x(t_0) = x^0$ boshlang'ich shartni qanoatlantiruvchi $x(t)$ yechimini quyidagicha yozish mumkin:

$$x(t) = X(t, t_0)x^0.$$

Endi (5.2.1) sistemaning $y(t_0) = y^0$ boshlang'ich shartni qanoatlantiruvchi yechimini $y(t)$ deb belgilaylik. U holda

$$y(t) = X(t, t_0)y^0$$

munosabat o'rinli bo'lishi ravshan. Ko'rinib turibdiki, $x(t)$ va $y(t)$ yechimlar ayirmasi uchun

$$x(t) - y(t) = X(t, t_0)(x^0 - y^0) \quad (5.2.4)$$

tenglik o'rinli bo'ladi. Oxirgi tenglikdan ko'rinib turibdiki, (5.2.1) sistema yechimini turg'unlikka tekshirishda, uning $X(t, t_0)$ -matritsanti asosiy ro'lni o'ynaydi.

5.2.1-teorema. Bir jinsli (5.2.1) differensial tenglamalar sistemasining $x(t)$ yechimi turg'un bo'lishi uchun, uning $X(t, t_0)$ fundamental matritsasi $[t_0, \infty)$ oraliqda chegaralangan bo'lishi zarur va yetarli.

5.2.2-teorema. Bir jinsli (5.2.1) differensial tenglamalar sistemasining $x(t)$ yechimi asimptotik turg'un bo'lishi uchun ushbu

$$\lim_{t \rightarrow \infty} |X(t, t_0)| = 0$$

munosabatning bajarilishi zarur va yetarli. Bu yerda $|\cdot|$ belgi $X(t, t_0)$ - matritsaning normasini bildiradi.

5.2.3-teorema. Bir jinsli (5.2.1) differensial tenglamalar sistemasining $x(t)$ yechimi turg'un bo'lmashligi uchun $X(t, t_0)$ matritsaning $[t_0, \infty)$ oraliqda chegaralanmagan bo'lishi zarur va yetarli.

Endi yuqoridagi teoremlarning isbotlarini keltiramiz.

Isbot (5.2.1-teorema) Zaruriyligi. Aytaylik, (5.2.1) differensial tenglamalar sistemasining $x = x(t)$ yechimi turg'un bo'lsin. U holda ixtiyoriy $\varepsilon > 0$ soni uchun shunday $\delta(\varepsilon) > 0$ soni topilib,

$$|x^0 - y^0| < \delta \quad (5.2.5)$$

tengsizlikning bajarilishidan ushbu

$$|x(t) - y(t)| < \varepsilon, \quad \forall t \geq t_0 \quad (5.2.6)$$

bahoning o'rinli bo'lishi kelib chiqadi. Avvalo (5.2.4) tenglikni quyidagicha yozib olamiz:

$$|x_i(t) - y_i(t)| = \left| \sum_{j=1}^n x_j^i(t)(x_j^0 - y_j^0) \right|, \quad i = \overline{1, n}. \quad (5.2.7)$$

Bu tenglikda quyidagi

$$x_1^0 = y_1^0, \dots, x_{k-1}^0 = y_{k-1}^0, x_k^0 \neq y_k^0, x_{k+1}^0 = y_{k+1}^0, \dots, x_n^0 = y_n^0$$

munosabatlardan foydalansak, ushbu

$$|x_k^i(t)| = \frac{|x_i(t) - y_i(t)|}{|x_k^0 - y_k^0|} \leq \overline{K}, \quad \forall t \geq t_0 \quad (5.2.8)$$

bahoga ega bo'lamiz. Bundan

$$|X(t, t_0)| \leq n\overline{K}, \quad t \geq t_0$$

kelib chiqadi. Bu esa $X(t, t_0)$ matritsaning $[t_0, \infty)$ oraliqda chegaralanganligini ko'rsatadi.

Yetarliligi. Aytaylik, $X(t, t_0)$ matritsa $[t_0, \infty)$ oraliqda chegaralangan, ya'ni shunday $M > 0$ soni mavjud bo'lib,

$$|X(t, t_0)| = \sum_{i,j=1}^n |x_i^j(t)| \leq M, \quad \forall t \geq t_0$$

baho o'rinli bo'lsin. Bu holda (5.2.1) sistemaning ixtiyoriy $x(t)$ yechimi uchun

$$|x(t) - y(t)| \leq |X(t, t_0)| \cdot |x^0 - y^0| \leq M |x^0 - y^0|, \quad \forall t \geq t_0$$

tengsizlik bajariladi. Ixtiyoriy $\forall \varepsilon > 0$ soni uchun $\delta > 0$ sonini $\delta = \varepsilon/M$ deb tanlasak, u holda

$$|x(t_0) - y(t_0)| = |x^0 - y^0| < \delta$$

tengsizligining bajarilishidan

$$|x(t) - y(t)| \leq M |x^0 - y^0| < M \cdot \frac{\varepsilon}{M} = \varepsilon$$

bahoning o'rinli bo'lishi kelib chiqadi. Bu esa (5.2.1) sistemaning $x = x(t)$ yechimini turg'unligini bildiradi. 5.2.1-teorema isbotlandi. ■

Isbot(5.2.2-teorema). Zaruriyligi. Aytaylik, (5.2.1) sistemaning $x = x(t)$ yechimi asimptotik turg'un bo'lsin, ya'ni

$$\lim_{t \rightarrow \infty} |x_i(t) - y_i(t)| = 0, \quad i = \overline{1, n}.$$

Bundan foydalanib, (5.2.8) tengsizlikda $t \rightarrow \infty$ da limitga o'tsak, undan

$$\lim_{t \rightarrow \infty} |X(t, t_0)| = 0$$

tenglik kelib chiqadi.

Yetarliligi. Faraz qilaylik, $\lim_{t \rightarrow \infty} |X(t, t_0)| = 0$ bo'lsin. U holda $X(t, t_0)$ – matritsa chegaralangan bo'lib, (5.2.1) sistemaning $x = x(t)$ yechimi turg'un bo'ladi. Bundan tashqari, (5.2.4) tenglikdan

$$\lim_{t \rightarrow \infty} |x(t) - y(t)| = \lim_{t \rightarrow \infty} |X(t, t_0)| \cdot |x^0 - y^0| = 0$$

hosil bo'ladi. Bu esa $x = x(t)$ yechimning asimptotik turg'un ekanligini ko'rsatadi. 5.2.2-teorema isbotlandi. ■

Isbot (5.2.3-teorema). Zaruriyligi. Faraz qilaylik, (5.2.1) sistemaning $x = x(t)$ yechimi turg'un bo'lmasin. U holda $X(t, t_0)$ matritsa $[t_0, \infty)$ oraliqda chegaralanmagan bo'ladi. Aks holda $X(t, t_0)$ matritsaning chegaralanganligidan $x = x(t)$ yechimning turg'unligi kelib chiqadi.

Yetarliligi. Faraz qilaylik, $X(t, t_0)$ matritsa $[t_0, \infty)$ oraliqda chegaralanmagan bo'lsin, ya'ni shunday monoton o'suvchi

$$t_0 \leq t_1 < t_2 < \dots < t_k < \dots$$

ketma-ketlik topilib,

$$\lim_{k \rightarrow \infty} |X(t_k, t_0)| = \infty$$

munosabat bajarilsin.

Endi (5.2.1) sistemaning

$$\bar{x}(t_0) = \bar{x}^0 = (\bar{x}_1^0, \bar{x}_2^0, \dots, \bar{x}_{m-1}^0, \bar{x}_m^0, \dots, \bar{x}_n^0)$$

boshlang'ich shartni qanoatlantiruvchi $\bar{x}(t)$ yechimini shunday tanlaymizki, natijada quyidagi

$$\bar{x}_1^0 = y_1^0, \bar{x}_2^0 = y_2^0, \dots, \bar{x}_{m-1}^0 = y_{m-1}^0, \bar{x}_m^0 \neq y_m^0, \bar{x}_{m+1}^0 = y_{m+1}^0, \dots, \bar{x}_n^0 = y_n^0$$

munosabatlar bajarilsin. Bunday tuzilgan $\bar{x}(t)$ yechim uchun ushbu

$$\bar{x}_j(t) - y_j(t) = x_j^m(t)(\bar{x}_m^0 - y_m^0)$$

tenglikning bajarilishi ravshan. Shuning uchun quyidagi

$$\lim_{k \rightarrow \infty} |\bar{x}_j(t_k) - y_j(t_k)| = \infty$$

munosabat o‘rinli. Bu esa $|\overline{x}_m^0 - y_m^0|$ ayirmaning moduli qanchalik kichik bo‘lganda ham $|\overline{x}_j(t) - y_j(t)|$ funksiya va $|\overline{x}(t_k) - y(t_k)|$ chegaralanmagandir. Shuning uchun (5.2.1) sistemaning $x = x(t)$ yechimi turg‘un bo‘lmaydi (noturg‘un bo‘ladi). 5.2.3-teorema isbotlandi. ■

Endi ushbu

$$\frac{dx}{dt} = a(t)x, \quad a(t) \in C[0, \infty) \quad (5.2.9)$$

ko‘rinishidagi bir jinsli chiziqli differensial tenglama yechimini turg‘unlikka tekshiramiz.

5.2.4-teorema. (5.2.9) differensial tenglama yechimining turg‘un bo‘lishi uchun quyidagi

$$\overline{\lim}_{t \rightarrow \infty} \int_0^t a(\tau) d\tau < +\infty \quad (5.2.10)$$

shartning bajarilishi zarur va yetarli.

Isbot. Zarurligi. Faraz qilaylik, (5.2.9) differensial tenglamaning yechimi turg‘un bo‘lsin. U holda (5.2.9) differensial tenglamaning

$$x(0) = x_0 \quad (5.2.11)$$

boshlang‘ich shartni qanoatlantiruvchi yechimi

$$x(t) = x_0 \exp \left\{ \int_0^t a(\tau) d\tau \right\} \quad (5.2.12)$$

formula orqali topiladi.

(5.2.9) differensial tenglamaning ushbu

$$y(0) = y_0 \quad (5.2.13)$$

boshlang‘ich shartni qanoatlantiruvchi yechimini $y(t)$ orqali belgilaylik. U holda $x(t)$ va $y(t)$ yechimlar ayirmasi uchun quyidagi

$$x(t) - y(t) = \exp \left\{ \int_0^t a(\tau) d\tau \right\} (x_0 - y_0) \quad (5.2.14)$$

tenglik o‘rinli bo‘ladi. (5.2.9) differensial tenglama yechimining turg‘unligidan, ya’ni ixtiyoriy $\forall \varepsilon > 0$ uchun shunday $\exists \delta = \delta(\varepsilon) > 0$ soni topilib, $|x_0 - y_0| < \delta$ tengsizlikning bajarilishidan $|x(t) - y(t)| < \varepsilon, \forall t \geq 0$ bahoning bajarilishi kelib chiqadi. Bundan va (5.2.14) tenglikdan

$$\exp \left\{ \int_0^t a(\tau) d\tau \right\} |x_0 - y_0| < \varepsilon, \quad \forall t > 0 \quad (5.2.15)$$

tengsizlikni olamiz. Bu tengsizlik bajarilishi uchun

$$\overline{\lim}_{t \rightarrow \infty} \int_0^t a(\tau) d\tau < +\infty$$

bo'lishi lozim.

Yetarligi. Aytaylik, (5.2.10) tengsizlik bajarilsin. U holda shunday $\exists K > 0$ soni topilib

$$\exp\left(\int_0^t a(\tau) d\tau\right) \leq K \quad (5.2.16)$$

o'rinli bo'ladi. Ixtiyoriy $\varepsilon > 0$ soni uchun $\delta > 0$ sonini $\delta = \frac{\varepsilon}{K}$ deb olamiz. U holda $x(t), y(t)$ yechimlarning $t=0$ nuqtadagi boshlang'ich shartlari $|x_0 - y_0| < \delta$ tengsizlikni qanoatlantirganda

$$|x(t) - y(t)| = \exp\left\{\int_0^t a(\tau) d\tau\right\} |x_0 - y_0| \leq K |x_0 - y_0| < \varepsilon$$

baho ixtiyoriy $\forall t \geq 0$ uchun bajariladi. Bu esa (5.2.9) differensial tenglamaning yechimi turg'un ekanligini bildiradi.

3-§ O'zgarmas koeffitsiyentli bir jinsli differensial tenglamalar sistemasi yechimining turg'unligi

Ushbu

$$\frac{dx}{dt} = Ax \quad (5.3.1)$$

ko'rinishidagi o'zgarmas koeffitsiyentli differensial tenglamalar sistemasini qaraylik. Bunda $A = \|a_{ij}\|$, $i, j = \overline{1, n}$ haqiqiy elementli o'zgarmas matritsa, $x(t) = (x_1(t), \dots, x_n(t))^T$ -noma'lum vektor funksiya. Bu paragrafda (5.3.1) differensial tenglamalar sistemasi $x(t) = 0$ yechimi turg'unligini o'rganamiz. Quyidagi

$$x(0) = x^0, x^0 = (x_1^0, x_2^0, \dots, x_n^0)^T \quad (5.3.2)$$

boshlang'ich shartni qanoatlantiruvchi (5.3.1) sistemaning $x(t)$ yechimi $t \in R$ da aniqlangan bo'lishi oldingi paragraflardan ma'lum. $\lambda_m, 1 \leq m \leq n$ orqali A matritsaning xos qiymatlarini, ya'ni

$$\det\|A - \lambda E\| = 0 \quad (5.3.3)$$

xarakteristik tenglama ildizlarini belgilaylik.

5.3.1-teorema. 1) Agar A matritsaning barcha $\lambda_j = \alpha_j + i\beta$, $i = \sqrt{-1}$, $j = \overline{1, n}$ xos qiymatlari uchun $\alpha_j = \operatorname{Re}\{\lambda_j\} < 0$, $1 \leq j \leq n$ shart bajarilsa,

u holda (5.3.1) differensial tenglamalar sistemasining $x(t) \equiv 0$ yechimi asimptotik turg'un bo'ladi.

2) Agar A matritsaning barcha $\lambda_j = \alpha_j + i\beta, i = \sqrt{-1}, j = \overline{1, n}$ xos qiymatlaridan birortasi uchun $\alpha_s = \operatorname{Re}\{\lambda_s\} \geq 0$ shart bajarilsa, u holda (5.3.1) differensial tenglamalar sistemasining $x(t) \equiv 0$ yechimi asimptotik turg'un bo'lmaydi.

Isbot. 1) Aytaylik, (5.3.3) xarakteristik tenglama $\lambda_j = \alpha_j + i\beta, i = \sqrt{-1}, j = \overline{1, n}$ ko'rinishidagi (umuman olganda karrali) ildizlarga ega bo'lsin. U holda (5.3.1) sistemaning ixtiyoriy yechimi uchun

$$x(t) = \sum_{j=1}^m e^{\alpha_j t} (P_j(t) \cos \beta_j t + Q_j(t) \sin \beta_j t) \quad (5.3.4)$$

formula o'rinli. Bu yerda $P_j(t), Q_j(t)$ vektor-funksiyalar bo'lib, ularning koordinatalari, darajasi λ_j xos qiymatlarining q_j karraligidan oshmaydigan ko'phadlar. Bunda $q_1 + q_2 + \dots + q_m = n$ munosabat o'rinli. (5.3.4) tenglikdan

$$\|x(t)\| \leq \sum_{j=1}^m e^{\alpha_j t} (\|P_j(t)\| + \|Q_j(t)\|) \quad (5.3.5)$$

baho kelib chiqadi. Teoremaning $\operatorname{Re} \lambda_j = \alpha_j < 0, j = \overline{1, m}$ shartiga asosan (5.3.5) munosabatdan

$$\lim_{t \rightarrow \infty} \|x(t)\| = 0 \quad (5.3.6)$$

ekanligini topamiz. Chunki, agar $\alpha < 0$ bo'lsa, u holda $P(t)e^{\alpha t} \rightarrow 0, t \rightarrow \infty$ o'rinli.

Bunda $P(t)$ -ko'phad. Yuqoridagi (5.3.6) munosabat (5.3.1) sistema $x(t) \equiv 0$ yechimining asimptotik turg'un ekanligini ko'rsatadi.

2) Aytaylik, (5.3.3) xarakteristik tenglama ildizlaridan birortasi ushbu

$$\lambda_s = \alpha + i\beta, \operatorname{Re} \lambda_s = \alpha > 0, (\beta \in R)$$

shartni qanoatlantirsin. U holda (5.3.1) sistema quyidagi

$$x(t) = e^{\alpha t} (a \cos \beta t + b \sin \beta t), a, b \in R^n, \|a\| + \|b\| \neq 0$$

ko'rinishidagi yechimga ega bo'ladi. Bundan $\|x(t)\| \not\rightarrow 0, (t \rightarrow \infty)$ ekanligi kelib chiqadi. Chunki $e^{\alpha t} \rightarrow \infty, (t \rightarrow \infty), \alpha > 0$. Bu esa $x(t) \equiv 0$ yechimining asimptotik turg'unligiga zid.

Endi, faraz qilaylik (5.3.3) xarakteristik tenglama ildizlaridan birortasi $\lambda_s = i\mu (\mu \in R)$ ko'rinishidagi, ya'ni $\operatorname{Re} \lambda_s = 0$ sondan iborat bo'lsin. U holda (5.3.1) sistema

$$x(t) = a \cos \mu t + b \sin \mu t \quad a, b \in R^n, \|a\| + \|b\| \neq 0$$

ko‘rinishdagi yechimga ega bo‘ladi. Bundan ko‘rinadiki, $\|x(t)\| \not\rightarrow 0, (t \rightarrow \infty)$.
 Bu esa $x(t) \equiv 0$ yechimining asimptotik turg‘unligiga zid. ■

Mustaqil yechish uchun mashqlar [8], §26, №886-894.

4-§. Turg‘unlikni birinchi yaqinlashish yordamida tekshirish

Aytaylik, bizga quyidagi

$$\frac{dx(t)}{dt} = f(x(t)) \quad (5.4.1)$$

ko‘rinishidagi muxtor differensial tenglamalar sistemasi berilgan bo‘lsin. Bu yerda $t \in R, x(t) = (x_1(t), \dots, x_n(t))^T, f(x) = (f_1(x), \dots, f_n(x))^T$ vektor-funksiyalar bo‘lib, $f(x)$ biror $G \subset R_x^n$ sohada uzluksiz differensiullanuvchi vektor-funksiya. Bundan tashqari $x(t) = 0$ nuqta (5.4.1) sistemaning muvozanat nuqtasi, ya’ni $f(0) = 0$ bo‘lsin. U holda $f(x)$ vektor-funksiyani $x = 0$ nuqta atrofida Teylor formulasidan foydalanib quyidagicha

$$f(x) = Ax + \bar{o}(|x|) \quad (5.4.2)$$

yo‘zish mumkin. Bunda qoldiq had $\bar{o}(|x|)$ Peano ko‘rinishida olingan:

$$A = \left\| \frac{\partial f_i(0)}{\partial x_j} \right\|, \quad i, j = \overline{1, n}; \quad \bar{o}(|x|) \rightarrow 0, \quad |x| = \sqrt{\sum_{j=1}^n |x_j|^2} \rightarrow 0.$$

5.4.1-teorema. Agar A matritsaning barcha $\lambda_j = \alpha_j + i\beta, j = \overline{1, m}, m \leq n$ xos qiymatlari

$$\alpha_j = \operatorname{Re} \lambda_j < 0, \quad j = \overline{1, n} \quad (5.4.3)$$

tengsizlikni qanoatlantirsa, u holda (5.4.1) nochiziqli muxtor sistemaning $x(t) \equiv 0$ yechimi asimptotik turg‘un bo‘ladi.

Isbot. Berilgan (5.4.1) muxtor sistemani $x = 0$ nuqtaning atrofida quyidagicha yo‘zish mumkin:

$$\frac{dx(t)}{dt} = Ax(t) + r(x). \quad (5.4.4)$$

Bu yerda

$$r(x) = \bar{o}(|x|). \quad (5.4.5)$$

Berilgan (5.4.1) muxtor differensial tenglamalar sistemasining ushbu

$$x(0) = x^0, x^0 = (x_1^0, x_2^0, \dots, x_n^0)^T \quad (5.4.6)$$

boshlang‘ich shartni qanoatlantiruvchi $x(t)$ yechimi $\forall t \in [0; \infty)$ oraliqda aniqlangan. Bu $x(t)$ yechimni

$$x(t) = e^{tA} x^0 + \int_0^t e^{(t-\tau)A} r[x(\tau)] d\tau \quad (5.4.7)$$

ko‘rinishida yozish mumkin. Ma’lumki, bir jinsli chiziqli differensial tenglamalar sistemasi uchun quyidagi

$$\frac{dx(t)}{dt} = Ax(t), \quad x(0) = x^0$$

Koshi masalasining $x(t, x^0)$ yechimini ushbu

$$x(t, x^0) = e^{tA} x^0 \quad (5.4.8)$$

ko‘rinishida yozish mumkin edi.

5.4.1-lemma. Agar A matritsaning barcha $\lambda_j = \alpha_j + i\beta_j$, $j = \overline{1, m}$, $m \leq n$ xos qiymatlari

$$\alpha_j = \operatorname{Re} \lambda_j < 0, \quad j = \overline{1, m}, \quad m \leq n$$

tengsizlikni qanoatlantirsa, u holda shunday $\exists \mu > 0, M > 0$ sonlari topilib,

$$\|e^{tA}\| \leq Me^{-\mu t} \quad (5.4.9)$$

baho o‘rinli bo‘ladi.

Isbot. Aytaylik, $\mu > 0$ sonini shunday tanlaymizki, natijada ushbu $\operatorname{Re} \lambda_j < -\mu < 0$ tengsizlik bajarilsin. U holda

$$\operatorname{Re} \lambda_j + \mu \leq -\nu, \quad \nu > 0, \quad j = \overline{1, m}, \quad m \leq n \quad (5.4.10)$$

o‘rinli bo‘ladi. Ma’lumki, e^{tA} matritsaning har bir $a_{ij}(t)$ elementi uchun quyidagi

$$a_{ij}(t) = \sum_{k=1}^m P_k^{(i,j)}(t) e^{\lambda_k t}, \quad i, j = \overline{1, n}$$

ko‘rinish o‘rinli bo‘ladi. Bu yerda $P_k^{(i,j)}(t)$ ko‘phad.

Endi (5.4.10) tengsizlikdan foydalanib

$$|P_k^{(i,j)}(t) e^{(\lambda_k + \mu)t}| = |P_k^{(i,j)}(t) e^{\operatorname{Re} \lambda_k + \mu)t}| \leq |P_k^{(i,j)}(t)| e^{-\nu t} \rightarrow 0, \quad t \rightarrow \infty$$

munosabatni olamiz. Bundan

$$|P_k^{(i,j)}(t) e^{(\lambda_k + \mu)t}| \leq c_{ij}, \quad \forall t > 0$$

kelib chiqadi. Bu esa o‘z navbatida

$$|P_k^{(i,j)}(t) e^{\lambda_k t}| \leq c_{i,j} e^{-\mu t}, \quad \forall t > 0 \quad (5.4.11)$$

ekanligini bildiradi. Endi (5.4.8) formuladan foydalanib quyidagi

$$|x(t, x^0)| \leq \|e^{tA}\| \cdot |x^0| = |x^0| \cdot \sqrt{\sum_{i,j=1}^n |a_{ij}(t)|^2} \leq |x^0| e^{-\mu t} m \left(\sum_{i,j=1}^n |c_{ij}|^2 \right)^{1/2} = Me^{-\mu t} \cdot |x^0|,$$

ya’ni

$$\|e^{tA}\| \leq Me^{-\mu t}, \quad \mu > 0$$

bahoni olamiz. ■

5.4.1-teoremani isbotlashda davom etamiz. Yuqoridagi (5.4.5) bahoda

$$r(x) = \overline{o}(|x|), \quad |x| \rightarrow 0$$

bo'lgani uchun, ixtiyoriy $\forall \varepsilon > 0$ soni uchun shunday $\exists \delta > 0$ soni topilib, $|x| < \delta$ bajarilganda

$$|r(x)| \leq \varepsilon |x|$$

bo'lishini inobatga olsak, (5.4.7) munosabatdan

$$\begin{aligned} |x(t)| &\leq |e^{tA} x^0| + \int_0^t |e^{(t-\tau)A} \cdot r(x(\tau))| d\tau \leq \|e^{tA}\| \cdot |x^0| + \int_0^t \|e^{(t-\tau)A}\| \cdot |r(x(\tau))| d\tau \leq \\ &\leq M e^{-\mu t} |x^0| + \varepsilon M \int_0^t e^{-\mu(t-\tau)} |x(\tau)| d\tau \end{aligned}$$

baho kelib chiqadi. Bu yerda ushbu

$$u(t) = e^{\mu t} |x(t)|$$

belgilashdan foydalansak,

$$u(t) \leq M |x^0| + \varepsilon M \int_0^t u(\tau) d\tau$$

tengsizlik hosil bo'ladi. $u(t)$ funksiyaga Gronoulla lemmasini qo'llasak,

$$u(t) \leq M |x^0| e^{\varepsilon M t}, \quad \forall t > 0$$

baho kelib chiqadi. Bundan esa

$$|x(t)| \leq M |x^0| e^{-(\mu - \varepsilon M)t}, \quad \forall t > 0$$

kelib chiqadi. Bu bahodan foydalanib, yetarli kichik $\varepsilon > 0$ larda (masalan $0 < \varepsilon < \frac{\mu}{M}$ bo'lganda)

$$|x(t)| \rightarrow 0, \quad t \rightarrow +\infty$$

munosabatni topamiz. Bu esa (5.4.1) sistema $x(t) \equiv 0$ yechimining asimptotik turg'un ekanligini ko'rsatadi. ■

5.4.2-teorema. Agar A matritsa $\operatorname{Re} \lambda > 0$ tengsizlikni qanoatlantiruvchi kamida bitta xos qiymatga ega bo'lsa, u holda (5.4.1) sistemaning $x(t) \equiv 0$ yechimi turg'un bo'lmaydi (noturg'un bo'ladi).

5.4.1-misol. Ushbu

$$\begin{cases} \dot{x}_1(t) = 4 \sin x_1 + \ln(1 + x_2), \\ \dot{x}_2(t) = x_1 + x_2 + x_1^2 x_2 \end{cases}$$

differensial tenglamalar sistemasining $x(t) = (x_1(t), x_2(t)) \equiv 0$ yechimini turg'unlikka tekshiring.

Yechish. Quyidagi

$$\sin x_1(t) \approx x_1(t), \quad \ln(1 + x_2(t)) \approx x_2(t)$$

munosabatlardan foydalanib,

$$\begin{cases} \dot{x}_1(t) = 4x_1 + x_2, \\ \dot{x}_2(t) = x_1 + x_2 \end{cases}$$

chiziqli differensial tenglamalar sistemasini tuzib olamiz. Endi ushbu

$$A = \begin{vmatrix} 4 & 1 \\ 1 & 1 \end{vmatrix}$$

matritsaning xos qiymatlarini aniqlaymiz:

$$\det\|A - \lambda E\| = \begin{vmatrix} 4 - \lambda & 1 \\ 1 & 1 - \lambda \end{vmatrix} = 0, \lambda^2 - 5\lambda + 3 = 0.$$

Bu kvadrat tenglamani yechib, $\lambda_1 = \frac{5 + \sqrt{13}}{2}$, $\lambda_2 = \frac{5 - \sqrt{13}}{2}$ xos qiymatlarni topamiz. Ko‘rinib turibdiki, $\lambda_{1,2} > 0$. Shuning uchun 5.4.2-teoremaga ko‘ra berilgan nochiziqli differensial tenglamalar sistemasining $x(t) \equiv 0$ yechimi turg‘un bo‘lmaydi (ya’ni noturg‘un bo‘ladi).

5.4.2-misol. Ushbu

$$\begin{cases} \dot{x}_1(t) = -\sin(x_1 - x_3)\cos x_2, \\ \dot{x}_2(t) = \sin^3 x_3 - x_2 - \sin x_3, \\ \dot{x}_3(t) = \operatorname{tg}(x_2 - x_3)\cos(x_1 - x_3) \end{cases}$$

nochiziqli differensial tenglamalar sistemasining $x(t) \equiv 0$ yechimini turg‘unlikka tekshiring

Yechish. Berilgan nochiziqli sistemaga mos keluvchi quyidagi

$$\begin{cases} \dot{x}_1(t) = -x_1 + x_3, \\ \dot{x}_2(t) = -x_2 - x_3, \\ \dot{x}_3(t) = x_2 - x_3 \end{cases}$$

chiziqli differensial tenglamalar sistemasini tuzib olamiz. Bunda $\sin(x_1 - x_3) \approx -x_1 - x_3$, $\sin x_3 \approx x_3$, $\operatorname{tg}(x_2 - x_3) \approx x_2 - x_3$, $\cos x_2 \approx 1$.

Quyidagi

$$A = \begin{vmatrix} -1 & 0 & 1 \\ 0 & -1 & -1 \\ 0 & 1 & -1 \end{vmatrix}$$

matritsaning xos qiymatlarini aniqlaymiz:

$$\det\|A - \lambda E\| = \begin{vmatrix} -1 - \lambda & 0 & 1 \\ 0 & -1 - \lambda & -1 \\ 0 & 1 & -1 - \lambda \end{vmatrix} = 0,$$

$$\lambda^3 + 3\lambda^2 + 4\lambda + 2 = 0, \lambda_1 = -1, \lambda_{2,3} = -1 \pm i.$$

Bu xos qiymatlarning barchasi uchun $\operatorname{Re}\{\lambda_j\} < 0$, $j=1,2,3$ tengsizlik bajariladi. Shuning uchun 5.4.1-teoremaga asosan berilgan noxizli differensial tenglamalar sistemasining $x(t) \equiv 0$ yechimi asimptotik turg'un bo'ladi.

Mustaqil yechish uchun mashqlar [21], §15, №793-796.

5-§. Turg'unlikni Lyapunov funksiyasi yordamida tekshirish

Ushbu

$$\frac{dx}{dt} = f(x), \quad x = x(t), \quad (5.5.1)$$

muxtor differensial tenglamalar sistemasini qaraylik. Bu yerda $f(x) = (f_1(x), f_2(x), f_3(x), \dots, f_n(x))^T$, $R^n \supset G$ sohada aniqlangan uzluksiz differensiallanuvchi vektor-funksiya, $x = x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T$ - nomalum vektor-funksiya, $t \in R$. Berilgan (5.5.1) muxtor sistemaga qo'yilgan $x(0) = x_0$, $x_0 \in G$

Koshi masalasining yechimini $x(t, x_0)$ orqali belgilaymiz.

Faraz qilaylik, $x(t) \equiv 0 \in G$ (5.5.1) sistemaning muvozanat nuqtasi, yani $f(0) = 0$ bo'lib, (5.5.1)-(5.5.2) Koshi masalasining $x(t, x_0)$ yechimi barcha $t > 0$, yani $\exists r > 0$, $|x_0| < r$, $\forall x_0 \in G$ lar uchun $x(t, x_0)$ yechim $\forall t > 0$ larda aniqlangan bo'lsin.

Aytaylik, $V(x)$, $x \in G \subset R^n$ sohada aniqlangan uzluksiz differensiallanuvchi, yani $V(x) \in C^1(G)$ vektor-funksiya bo'lsin. Bundan tashqari $U \subset G$ -(5.5.1) muxtor sistema $x \equiv 0$ muvozanat nuqtasining biror atrofi bo'lsin.

5.5.1-ta'rif. $V(x)$ vektor-funksiyaning (5.5.1) muxtor sistema bo'yicha to'liq hosilasi deb ushbu $(\operatorname{grad} V(x), f(x))$, $x \in G$ skalyar ko'paytmaga aytiladi va quyidagicha belgilanadi:

$$\dot{V}(x) = \frac{dV}{dt} = (\operatorname{grad} V(x), f(x)) = \sum_{i=1}^n \frac{\partial V}{\partial x_i} f_i(x_1, x_2, x_3, \dots, x_n), \quad x \in G.$$

5.5.2-ta'rif. Quyidagi

1) $V(x) > 0$, $\forall x \in U \setminus \{0\}$, $V(0) = 0$,

2) $\frac{dV(x)}{dt} = (\operatorname{grad}(V(x), f(x))) \leq 0$, $\forall x \in U$

shartlarni qanoatlantiruvchi $V(x) \in C^1(U)$ vektor-funksiyaga Lyapunov funksiyasi deyiladi.

5.5.1-lemma. Faraz qilaylik, $x \in U_1 \subset U, U_1 - x = 0$ nuqtaning biror atrofi bo'lsin. U holda $|x| \rightarrow 0$ bo'lishi uchun $V(x) \rightarrow 0$ bo'lishi zarur va yetarli.

Isbot. Agar $|x| \rightarrow 0$ bo'lsa, u holda Lyapunov funksiyasining ta'rifidan $V(x) \rightarrow 0$ ekanligi kelib chiqadi. Aytaylik, $V(x) \rightarrow 0$ bo'lsin, u holda $|x| \rightarrow 0$ bo'lishini ko'rsatamiz. Teskarisini, ya'ni $V(x) \rightarrow 0$ bo'lganda $|x| \not\rightarrow 0$ deb faraz qilaylik. U holda shunday $\exists x_k \in U_1$ ketma ketlik va $\exists \delta > 0$ soni topilib, $|x_k| \geq \delta$ tengsizlik bajariladi. Ushbu $|x| < a$ sharni U_1 da joylashadigan qilib $a > 0$ sonini tanlaymiz. Quyidagi

$$V_0 = \inf_{\delta \leq |x| \leq a} V(x)$$

belgilashni kiritaylik. Ushbu $\{x \in U_1; \delta \leq |x| \leq a\}$ to'plam kompakt bo'lgani uchun, shunday $\exists \bar{x} \in U_1, \delta \leq |\bar{x}| \leq a$ topilib, $V(\bar{x}) = V_0$ o'rinli bo'ladi. Bundan $V_0 > 0$ va $V(x_k) \geq V_0 > 0$ kelib chiqadi. Bu esa ziddiyat. ■

5.5.1-teorema (Lyapunov). Agar $x(t) \equiv 0$ muvozanat nuqtaning biror U atrofida (5.5.1) sistema uchun $V(x)$ Lyapunov funksiyasi mavjud bo'lsa, u holda (5.5.1) sistemasining $x(t) \equiv 0$ yechimi Lyapunov ma'nosida turg'un bo'ladi.

Isbot. $\varepsilon > 0$ sonini shunday tanlaymizki, $|x| \leq \varepsilon$ shar ushbu U atrofda joylashsin. Quyidagi

$$V_\varepsilon = \inf_{|x| \leq \varepsilon} V(x)$$

belgilashni kiritamiz. U holda $V_\varepsilon > 0$ o'rinli. Chunki, $V(x)$ -musbat aniqlangan. Bundan tashqari $0 < \delta < \varepsilon$ tengsizlikni qanoatlantiruvchi $\delta > 0$ soni topilib, $V(x) < V_\varepsilon$ tengsizlik barcha $\forall x, |x| < \delta$ larda bajariladi. Chunki $V(0) = 0$ va $V(x)$ -uzluksiz, ya'ni

$$\lim_{x \rightarrow 0} V(x) = V(0) = 0; \forall \varepsilon > 0, \exists \delta > 0, |x| < \delta, V(x) < \varepsilon$$

o'rinli. Bunda $\varepsilon = V_\varepsilon$ deb $V(x) < V_\varepsilon$ ni olamiz. Endi (5.5.1) sistemaning $|x_0| < \delta$ tengsizlikni qanoatlantiruvchi boshlang'ich shartlardagi ixtiyoriy $x(t, x_0)$ yechimi uchun $|x(t, x_0)| < \varepsilon, \forall t > 0$ bahoni o'rinli bo'lishini ko'rsatamiz. Bu esa o'z navbatda $x(t) = 0$ yechimning Lyapunov ma'nosida turg'unligini bildiradi. Teskarisini faraz qilaylik. U holda shunday $T > 0$ soni topilib, $|x(T, x_0)| = \varepsilon$ va $|x(t, x_0)| < \varepsilon, 0 \leq t \leq T$ da bajariladi. Chunki Lyapunov funksiyasining tarifiga ko'ra $\dot{V}(x) \leq 0, \forall x \in U$ bo'lgani uchun $V(x(t, x_0)), t \in [0, T]$ oraliqda o'smaydigan funksiya bo'ladi. Bunda $V(x_0) < V_\varepsilon$ ekanligini inobatga olsak,

$$V(x(T, x_0)) \leq V(x_0) < V_\varepsilon$$

o‘rinli. Bu esa $T > 0$ sonini tanlanishiga va V_ε ning aniqlanishiga zid. Shuning uchun farazimiz noto‘g‘ri bo‘lib, $x(t) = 0$ muvozanat nuqta Lyapunov ma‘nosida turg‘un bo‘ladi.

5.5.1-izoh. Bu teorema xususan, $\dot{V}(x) \equiv 0, x \in U$ bo‘lgan holda ham o‘z kuchini saqlaydi. Bu holda $V(x)$ - Lyapunov funksiyasi $x = 0$ da qat’iy minimumga ega.

5.5.2-teorema (Lyapunov). Agar $x(t) = 0$ muvozanat nuqtasining biror U atrofida (5.5.1) sistema uchun shunday $V(x)$ -Lyapunov funksiyasi mavjud bo‘lib, $\dot{V}(x) < 0 \quad \forall x \in U \setminus \{0\}$ tengsizlikni qanoatlantirsa, u holda (5.5.1) muxtor sistemaning $x(t) \equiv 0$ yechimi asimptotik turg‘un bo‘ladi.

Isbot. 5.5.1-teoremaning shartlari bajarilgani uchun $x(t) \equiv 0$ yechim Lyapunov ma‘nosida turg‘un bo‘ladi, ya‘ni $\forall \varepsilon > 0$ soni uchun $\exists \delta = \delta(\varepsilon) > 0$ soni topilib, $|x_0| < \delta$ tengsizligini qanoatlantiruvchi barcha x_0 lar uchun $\dot{x}(t) = f(t), x(0) = x_0$ Koshi masalasining har bir $x(t, x_0)$ yechimi uchun $|x(t, x_0)| < \varepsilon, \forall t > 0$ baho o‘rinli bo‘ladi. Yuqoridagi 5.5.1-lemmaga asosan, ushbu $|x_0| < \delta$ tengsizlikning bajarilishidan $V[x(t, x_0)] \rightarrow 0, t \rightarrow +\infty$ bo‘lishini ko‘rsatamiz. Chunki $\dot{V}(x) < 0, x \in U$ bo‘lgani uchun $V[x(t, x_0)] - t$ o‘zgaruvchi bo‘yicha qat’iy kamayuvchidir. Aytaylik, $\lim_{t \rightarrow +\infty} V[x(t, x_0)] = A$ bo‘lsin. U holda $A = 0$ ekanligini ko‘rsatamiz. Faraz qilaylik, $A > 0$ bo‘lsin. $V[x(t, x_0)] \geq 0$ bo‘lgani uchun 5.5.1-lemmaga ko‘ra shunday $\varepsilon_1 > 0$ soni topilib,

$$\varepsilon_1 \leq |x(t, x_0)| \leq \varepsilon_2 < \varepsilon, \forall t > 0$$

bajariladi. Quyidagi

$$V_1 = \inf_{\varepsilon_1 \leq |x| \leq \varepsilon_2} \{-V(x)\}$$

belgilashni kiritaylik. U holda $V_1 > 0$, chunki $\dot{V}(x) \leq -V_1$, u holda bu tengsizlikni trayektoriya bo‘yicha $t = 0$ dan t gacha integrallab,

$$V[x(t, x_0)] - V(x_0) \leq -V_1 t$$

munosabatni topamiz. Bundan $V[x(t, x_0)] \rightarrow -\infty, t \rightarrow +\infty$ kelib chiqadi. Bu esa $V[x(t, x_0)] \geq 0$ munosabatga zid. Hosil bo‘lgan ziddiyat farazimizning noturg‘unligini ko‘rsatadi. Shuning uchun $A = 0$, ya‘ni $\lim_{t \rightarrow +\infty} V[x(t, x_0)] = 0$. ■

5.5.1-misol. Ushbu differensial tenglamalar sistemasining $x(t) = (x_1(t), x_2(t)) \equiv 0$ yechimini turg‘unlikka tekshiring:

$$\begin{cases} \dot{x}_1 = x_2 + 3x_1^2 x_2^2 - 4x_1^5, \\ \dot{x}_2 = -x_1 - x_2^3 + x_1^3 x_2. \end{cases}$$

Yechish. Berilgan sistemaga mos keluvchi Lyapunov funksiyasi sifatida ushbu

$$V(x) = V(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2)$$

kvadratik formani olamiz. Endi $V(x)$ funksiyaning berilgan sistema bo'yicha hosilasini hisoblaymiz:

$$\begin{aligned} \frac{dV}{dt} &= x_1 \frac{dx_1}{dt} + x_2 \frac{dx_2}{dt} = x_1(x_2 + 3x_1^2x_2^2 - 4x_1^5) + x_2(-x_1 - x_2^3 + x_1^3x_2) = \\ &= -(2x_1^3 - x_2^2)^2 \leq 0. \end{aligned}$$

Demak, $\frac{dV}{dt} \leq 0$ bo'lib, $V(0) = 0$ bo'lgani sababli $V(x)$ -berilgan sistemaning Lyapunov funksiyasi bo'ladi. Shuning uchun 5.5.1-teoremaga ko'ra $x(t) \equiv 0$ yechim Lyapunov ma'nosida turg'un bo'ladi.

5.5.2-misol. Ushbu

$$\dot{x} = \sin x - x$$

differensial tenglamaning $x(t) \equiv 0$ yechimini turg'unlikka tekshiring.

Yechish. Quyidagi

$$V(x) = x^2$$

kvadrat funksiyaning qaraylik. Endi uning berilgan tenglama bo'yicha hosilasini hisoblaymiz:

$$\frac{dV}{dt} = \frac{d}{dt}(x^2(t)) = 2x(t) \frac{dx}{dt} = 2x(\sin x - x) < 0, \quad x \neq 0.$$

Demak ikkinchi teoremaga ko'ra $x(t) \equiv 0$ yechim asimptotik turg'un bo'ladi.

5.5.3-misol. Ushbu differensial tenglamalar sistemasining $x(t) = (x_1(t), x_2(t)) \equiv 0$ yechimini turg'unlikka tekshiring:

$$\begin{cases} \dot{x}_1(t) = 2x_2^3 - x_1^5, \\ \dot{x}_2(t) = -x_1 - x_2^3. \end{cases}$$

Yechish. Quyidagi

$$V(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^4)$$

musbat aniqlangan funksiyaning qaraylik. Endi $V(x)$ funksiyaning berilgan sistema bo'yicha hosilasini hisoblaymiz:

$$\frac{dV}{dt} = x_1 \frac{dx_1}{dt} + 2x_2^3 \frac{dx_2}{dt} = x_1(2x_2^3 - x_1^5) + 2x_2^3(-x_1 - x_2^3) = -x_1^6 - 2x_2^6 < 0,$$

$$V(0,0) = 0.$$

Demak $x(t) = (x_1(t), x_2(t)) = 0$ yechim 5.5.2-teoremaga ko'ra asimptotik turg'un bo'lar ekan.

5.5.4-misol. Ushbu differensial tenglamalar sistemasining $x(t) = (x_1(t), x_2(t)) \equiv 0$ yechimini turg'unlikka tekshiring:

$$\begin{cases} \dot{x}_1 = -x_2 - x_1^3, \\ \dot{x}_2 = x_1 - x_2^3 \end{cases}$$

Yechish. Berilgan sistema uchun Lyapunov funksiyasini ushbu

$$V(x_1, x_2) = x_1^2 + x_2^2$$

ko‘rinishda olamiz va uning sistema bo‘yicha olingan hosilasini hisoblaymiz:

$$\frac{dV}{dt} = 2x_1 \frac{dx_1}{dt} + 2x_2 \frac{dx_2}{dt} = 2x_1(-x_2 - x_1^3) + 2x_2(x_1 - x_2^3) = -2x_1^4 - x_2^4 < 0,$$

$$V(x_1, x_2) > 0, |x| > 0; V(0,0) = \dot{V}(0,0) = 0.$$

Demak, 5.5.2-teoremaga ko‘ra $x(t) \equiv 0$ yechim asimptotik turg‘un bo‘lar ekan.

5.5.5-misol. Ushbu differensial tenglamalar sistemasining $x(t) = (x_1, x_2) \equiv 0$ yechimini turg‘unlikka tekshiring:

$$\begin{cases} \dot{x}_1 = -x_1^3, \\ \dot{x}_2 = -x_1^2(x_1 + \frac{1}{2}x_2). \end{cases}$$

Yechish. Berilgan sistema uchun Lyapunov funksiyasini

$$V(x_1, x_2) = x_1^2 + 2x_2^2$$

ko‘rinishda izlaymiz. Endi $V(x_1, x_2)$ funksiyasining berilgan sistema bo‘yicha hosilasini hisoblaymiz:

$$\begin{aligned} \frac{dV}{dt} &= 2x_1 \frac{dx_1}{dt} + 4x_2 \frac{dx_2}{dt} = 2x_1(-x_1^3) + 4x_2(-x_1^3 - \frac{1}{2}x_1^2x_2) = -2x_1^4 - 2x_1^2x_2^2 - 4x_1^3x_2 = \\ &= -2x_1^2(x_1^2 + x_2^2 + 2x_1x_2) = -2x_1^2(x_1 + x_2)^2 \leq 0, V(0,0) = 0, \dot{V}(0,0) = 0. \end{aligned}$$

Demak, 5.5.1-teoremaga ko‘ra $x(t) = (x_1(t), x_2(t)) \equiv 0$ yechim Lyapunov ma‘nosida turg‘un ekan.

5.5.6-misol. Ushbu differensial tenglamalar sistemasining $x(t) = (x_1(t), x_2(t)) \equiv (0,0)$ muvozanat nuqtasini turg‘unlikka tekshiring:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 \end{cases}$$

Yechish. Bu misolda Lyapunov funksiyasini

$$V(x_1, x_2) = x_1^2 + x_2^2$$

ko‘rinishda tanlaymiz va uning berilgan sistema bo‘yicha to‘liq hosilasini hisoblaymiz:

$$\frac{dV}{dt} = 2x_1 \frac{dx_1}{dt} + 2x_2 \frac{dx_2}{dt} = 2x_1(x_2) + 2x_2(-x_1) = 2x_1x_2 - 2x_1x_2 = 0,$$

$$V(0,0) = 0.$$

Demak, $(0,0)$ muvozanat nuqta Lyapunov ma'nosida turg'un, lekin asimptotik turg'un emas (noturg'un). Chunki 5.5.2-teoremaning shartlari bajarilmaydi.

5.5.7-misol. Ushbu differensial tenglamalar sistemasining $x(t) \equiv 0$ yechimini turg'unlikka tekshiring:

$$\begin{cases} \frac{dx_1}{dt} = -x_1 + x_1^3 x_2^2, \\ \frac{dx_2}{dt} = -2x_2 + x_1^2 x_2^5 \end{cases}$$

Yechish. Berilgan differensial tenglamalar sistemasiga mos keluvchi Lyapunov funksiyasi sifatida quyidagi $V = x_1^2 + x_2^2$ kvadratik formani olamiz. Endi $V(x) = V(x_1, x_2)$ funksiyaning berilgan sistema bo'yicha to'liq hosilasini hisoblaymiz:

$$\begin{aligned} \frac{dV}{dt} &= 2x_1 \frac{dx_1}{dt} + 2x_2 \frac{dx_2}{dt} = 2x_1(-x_1 + x_1^3 x_2^2) + 2x_2(-2x_2 + x_1^2 x_2^5) = \\ &= -2x_1^2 - 4x_2^2 + 2x_1^4 x_2^2 + 2x_1^2 x_2^6. \end{aligned}$$

Bundan ko'rinadiki, $x(t) \equiv 0$ nuqtaning biror atrofida $W = \frac{dV}{dt}$ funksiyasining

ishorasi ushbu $-2x_1^2 - 4x_2^2$ kvadratik formaning ishorasi bilan aniqlanadi.

Chunki $2x_1^4 x_2^2 + 2x_1^2 x_2^6 > 0$. Shuning uchun $W = \frac{dV}{dt} < 0$, $W = \frac{dV}{dt} \Big|_{x(t)=0} = 0$.

Demak, Lyapunov teoremasining sharti bajariladi. Bunga ko'ra, $x(t) \equiv 0$ yechim asimptotik turg'un bo'ladi.

5.5.3-teorema. (Barbashin, Krasovskiy). Aytaylik, $|x| < \rho$ sharda aniqlangan quyidagi

$$V(x) \in C^1, V(0) = 0, V(x) > 0 \ (x \neq 0), \frac{dV}{dt} \Big|_{x(t)=0} \leq 0$$

shartlarni qotlantiruvchi $V(x)$ funksiya mavjud bo'lib, ushbu

$N = \left\{ x(t) : \frac{dV}{dt} \Big|_{x(t)=0} = 0 \right\}$ to'plamga faqat $x(t) = 0$ traektoriya qarashli bo'lsin.

U holda (5.5.1) muxtor sistemaning $x(t) \equiv 0$ yechimi asimptotik turg'un bo'ladi.

Teoremani isbotlash o'rniga quyidagi misolni keltiramiz.

5.5.8-misol. Ushbu differensial tenglamalar sistemaning $x(t) = (x_1(t), x_2(t)) \equiv 0$ yechimini turg'unlikka tekshiring:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -x_1^3 - ax_2, \ a = const \end{cases}$$

Yechish. 1-hol. $a = 0$ bo'lsin. U holda berilgan differensial tenglamalar sistemasidan

$$\frac{dx_2}{dx_1} = -\frac{x_1^3}{x_2}$$

tenglamani topamiz. Bu tenglamani integrallab,

$$2x_2^2(t) + x_1^4(t) = c, \quad c = \text{const}$$

munosabatni hosil qilamiz. Endi quyidagi

$$V(x) = 2x_2^2 + x_1^4$$

funksiyani qaraylik va uning berilgan muxtor sistema bo'yicha olingan hosilasini hisoblaylik:

$$\frac{dV}{dt} = 4x_2 \frac{dx_2}{dt} + 4x_1^3 \frac{dx_1}{dt} = 4x_2(-x_1^3) + 4x_1^3(x_2) = -4x_2x_1^3 + 4x_1^3x_2 = 0.$$

Demak, 5.5.1-teoremaga ko'ra, $x(t) = 0$ yechim Lyapunov ma'nosida turg'un bo'ladi. Ammo $x(t) = 0$ yechim asimptotik turg'un bo'lmaydi. Chunki, berilgan sistemaning ixtiyoriy nolmas yechimi uchun

$$x_1^4(t) + 2x_2^2(t) = c, \quad c = \text{const}$$

tenglik o'rinli. Bunda $t \rightarrow \infty$ da $x_1^4(t) + 2x_2^2(t) = c \rightarrow 0$.

2-hol. $a > 0$ bo'lsin. Bu holda ham yuqoridagi

$$V(x_1, x_2) = 2x_2^2(t) + x_1^4(t)$$

funksiyani olib, uning berilgan sistema bo'yicha hosilasini hisoblaymiz:

$$\begin{aligned} \frac{dV}{dt} &= 4x_2^2 \frac{dx_2}{dt} + 4x_1^3 \frac{dx_1}{dt} = 4x_2^2(-x_1^3 - ax_2) + 4x_1^3(x_2) = \\ &= -4x_2^2x_1^3 - 4ax_2^3 + 4x_1^3x_2 = -4ax_2^3 \leq 0. \end{aligned}$$

Bu yerda tenglik $x_2(t) = 0$ to'g'ri chiziqda bajariladi. Bu to'g'ri chiziqda $\dot{x}_2(t) = -x_1^3 \neq 0, x \neq 0$. Demak 5.5.3-teoremaga ko'ra $x(t) = 0$ yechim asimptotik turg'un bo'ladi.

3-hol. $a < 0$ bo'lsin. Bu holda

$$\frac{dV}{dt} = -4ax_2^3 \geq 0$$

munosabat o'rinli bo'lgani uchun hozircha $x(t) = 0$ yechimni turg'unlikka tekshira olmaymiz. Keyinchalik $x(t) = 0$ yechimning turg'un emasligi ko'rsatiladi.

5.5.4-teorema (Chitaev). Aytaylik $x \equiv 0$ muvozanat nuqtaning U atrofida aniqlangan ushbu

$$V(x) \in C^1(U), \quad V(0) = 0, \quad V(x) \geq 0 \quad \text{va} \quad \dot{V}(x) = (\text{grad}V(x), f(x)), \quad \dot{V}(x) \geq 0$$

munosabatlarni qanoatlantiruvchi uzluksiz differensiallanuvchi funksiya mavjud bo'lsin. Agar shunday $A > 0$ soni va U atrofda joylashgan U_0 - soha

mavjud bo‘lib, $V(x) \geq A$, $\forall x \in U_0$ tengsizlik bajarilsa, u holda $x \equiv 0$ yechim noturg‘un bo‘ladi.

5.5.9-misol. Ushbu differensial tenglamalar sistemasining $x(t) = (x_1(t), x_2(t)) = 0$ yechimini turg‘unlikka tekshiring:

$$\begin{cases} \dot{x}_1 = x_1 x_2^4 \\ \dot{x}_2 = x_2 x_1^2 \end{cases}$$

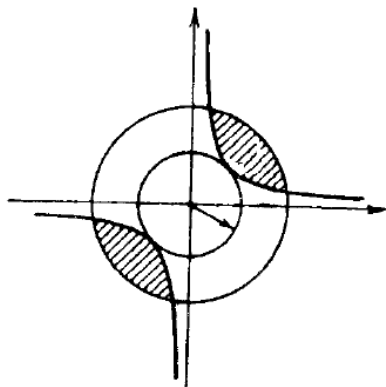
Yechish. Quyidagi

$$V(x_1, x_2) = x_1 x_2$$

funksiyani qaraylik va uning berilgan sistema bo‘yicha to‘liq hosilasini hisoblaylik:

$$\frac{dV}{dt} = x_2 \frac{dx_1}{dt} + x_1 \frac{dx_2}{dt} = x_2 (x_1 x_2^4) + x_1 (x_2 x_1^2) = x_1 x_2^5 + x_2 x_1^3 = x_1 x_2 (x_2^4 + x_1^4).$$

Endi $x = 0$ muvozanat nuqtaning U atrofi bilan $x_1 \cdot x_2 = A$ giperbola ajratgan U_0 - sohani chizmada ko‘rsatish mumkin:



6-chizma

Ko‘rinib turibdiki, $x_1^2 + x_2^2 \geq 2A$, $|x| = \sqrt{x_1^2 + x_2^2}$, $|x| \geq \sqrt{2A}$, $\forall x \in U_0$. Bundan va lemmadan $\exists \varepsilon > 0$ soni topilib, $x_1^4 + x_2^4 \geq \varepsilon$, ya’ni $V(x) \geq A\varepsilon > 0$, $x \in U_0$ bajarilishi kelib chiqadi. Demak, Chitaev teoremasiga ko‘ra, $x(t) = 0$ yechim noturg‘un bo‘ladi.

5.5.10-misol. Ushbu differensial tenglamalar sistemasining $x(t) = (x_1(t), x_2(t)) = (0, 0)$ yechimini turg‘unlikka tekshiring:

$$\begin{cases} \dot{x}_1 = x_1^3 + 2x_1 x_2^2, \\ \dot{x}_2 = x_1 x_2 \end{cases}$$

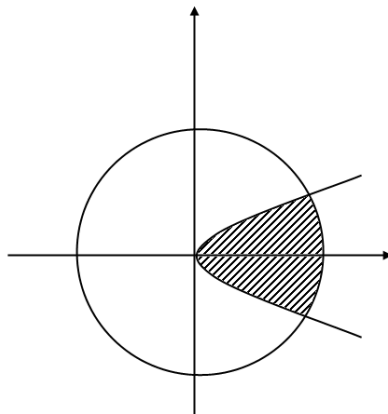
Yechish. Quyidagi

$$V(x_1, x_2) = x_1 - x_2^2$$

funksiyani olib, uning berilgan sistema bo‘yicha to‘liq hosilasini hisoblaymiz:

$$\frac{dV}{dt} = \frac{dx_1}{dt} - 2x_2 \frac{dx_2}{dt} = x_1^3 + 2x_1 x_2^2 - 2x_2 (x_1 \cdot x_2) = x_1^3 + 2x_1 x_2^2 - 2x_1 x_2^2 = x_1^3 \geq 0.$$

$V(x) = V(x_1, x_2) > 0$ da $x_1 > x_2^2$ tengsizligini qanoatlantiruvchi nuqtalar to'plami U_0 sohani beradi.



7-chizma

$$\partial U_0 = \gamma_0 \cup \gamma, V(x_1, x_2) \Big|_{\substack{x \in \gamma_0 \\ x_1 = x_2^2}} = (x_1 - x_2^2) \Big|_{x_1 = x_2^2} = 0,$$

$$V(x_1, x_2) \Big|_{x \in U_0} = (x_1 - x_2^2) \Big|_{x_1 > x_2^2} > 0.$$

Demak, Chitaev teoremasiga ko'ra, $x(t) = (x_1(t), x_2(t)) = 0$ yechim noturg'un bo'ladi.

Mustaqil yechish uchun mashqlar [8], §27, №895-908.

6-§. n -tartibli chiziqli differensial tenglama yechimini turg'unlikka tekshirish

Quyidagi

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} y' + a_n y = f(t), \quad (a_0 \neq 0), \quad t > t_0, \quad (5.6.1)$$

$$\vec{y}(t_0) = \vec{\varphi}_0 \quad (5.6.2)$$

Koshi masalasining $y = \varphi(t)$ yechimini qaraylik. Bu yerda

$$\vec{y}(t_0) = (y(t_0), y'(t_0), \dots, y^{(n-1)}(t_0))^T,$$

$$\vec{\varphi}_0 = (\varphi_0^0, \varphi_1^0, \dots, \varphi_{n-1}^0)^T, \quad \varphi_j^0 = \text{const}, \quad j = \overline{1, n-1}.$$

Ushbu

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} y' + a_n y = f(t), \quad t > t_0, \quad (5.6.3)$$

$$\vec{y}(t_0) = \vec{y}_0, \quad \vec{y}_0 = (y_0^0, y_1^0, \dots, y_{n-1}^0)^T \quad (5.6.4)$$

Koshi masalasining yechimini $y = y_\delta(t)$ orqali belgilaylik.

5.6.1-ta'rif. Agar $\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0$ soni topilib, ushbu

$$|\vec{y}_0 - \vec{\varphi}_0| < \delta \quad (5.6.5)$$

tengsizlikni qanoatlantiruvchi barcha $\forall \bar{y}_0$ lar uchun (5.6.3)-(5.6.4) Koshi masalasining $y_\delta(t)$ va $y = \varphi(x)$ yechimlari ushbu

$$|y_\delta(t) - \varphi(t)| < \varepsilon, \forall t > t_0 \quad (5.6.6)$$

bahoni qanoatlantirsa, (5.6.1)-(5.6.2) Koshi masalasining $y = \varphi(t)$ yechimiga Lyapunov ma'nosida turg'un deyiladi.

5.6.1-lemma. (5.6.1) differensial tenglama $y = \varphi(t)$ yechimining turg'unligi, ushbu

$$a_0 z^{(n)} + a_1 z^{(n-1)} + \dots + a_{n-1} z' + a_n z = 0 \quad (5.6.7)$$

bir jinsli tenglama $z(t) \equiv 0$ nol yechimining turg'unligiga ekvivalent.

Isbot. Ushbu

$$z_\delta(t) = y_\delta(t) - \varphi(t)$$

belgilashni kiritaylik. Bunda $y_\delta(t)$ quyidagi

$$\begin{aligned} a_0 y_\delta^{(n)} + a_1 y_\delta^{(n-1)} + \dots + a_{n-1} y_\delta' + a_n y_\delta &= f(t), \quad t > t_0, \\ y_\delta(t_0) = \bar{y}_0, \quad \bar{y}_0 &= (y_0^0, y_1^0, \dots, y_{n-1}^0)^T, \quad y_j^0 = \text{const}, \quad j = \overline{1, n-1}, \\ y_\delta(t_0) &= (y_\delta(t_0), y_\delta'(t_0), \dots, y_\delta^{(n-1)}(t_0))^T \end{aligned} \quad (5.6.8)$$

Koshi masalasining yechimidan iborat, $\varphi(t)$ esa ushbu

$$\begin{aligned} a_0 \varphi^{(n)} + a_1 \varphi^{(n-1)} + \dots + a_{n-1} \varphi' + a_n \varphi &= f(t), \quad t > t_0 \\ \bar{\varphi}(t_0) = \bar{\varphi}_0, \quad \bar{\varphi}_0 &= (\varphi_0^0, \varphi_1^0, \dots, \varphi_{n-1}^0)^T, \quad \varphi_j^0 = \text{const}, \quad j = \overline{1, n-1} \end{aligned} \quad (5.6.9)$$

Koshi masalasining yechimini ifodalaydi. Bu (5.6.8) va (5.6.9) munosabatlarni mos ravishda ayirib, quyidagi

$$\begin{aligned} a_0 z_\delta^{(n)}(t) + a_1 z_\delta^{(n-1)}(t) + \dots + a_{n-1} z_\delta'(t) + a_n z_\delta(t) &= 0, \quad t > t_0, \\ \bar{z}_\delta(t_0) = \bar{z}_0, \quad \bar{z}_0 &= \bar{y}_0 - \bar{\varphi}_0 \end{aligned} \quad (5.6.10)$$

bir jinsli tenglamaga qo'yilgan Koshi masalasini hosil qilamiz.

Yuqoridagi $y = \varphi(t)$ yechim turg'unligining ta'rifi $z_\delta(t)$ yechimga nisbatan quyidagicha bayon qilinadi: $\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0, |\bar{z}_0| < \delta$ tengsizlikni qanoatlantiruvchi barcha \bar{z}_0 lar uchun (5.6.10) Koshi masalasining yechimi ushbu

$$|z_\delta(t)| < \varepsilon, \forall t > t_0$$

bahoni qanoatlantiradi. Bu esa (5.6.7) bir jinsli tenglama $z(t) \equiv 0$ yechimining turg'unligini anglatadi.

5.6.1-natija. (5.6.1) ko'rinishdagi bir jinsli bo'lmagan n -tartibli chizikli differensial tenglama yechimini turg'unligini o'rganish o'rniga (5.6.7) ko'rinishdagi bir jinsli chizikli differensial tenglamaning nol yechimini turg'unligini o'rganish yetarli.

5.6.2-ta'rif. Agar (5.6.1) differensial tenglamaning $y = \varphi(t)$ yechimi Lyapunov ma'nosida turg'un bo'lib,

$$\lim_{t \rightarrow +\infty} (y_\delta(t) - \varphi(t)) = 0$$

munosabat bajarilsa, $y = \varphi(t)$ yechimga asimptotik turg'un yechim deyiladi.

5.6.2-lemma. (5.6.1) differensial tenglama $y = \varphi(t)$ yechimining asimptotik turg'un bo'lishi (5.6.7) bir jinsli tenglama $z(t) = 0$ yechimining asimptotik turg'un bo'lishiga ekvivalent.

Isbot. Ushbu $z_\delta(t) = y_\delta(t) - \varphi(t)$ almashtirishdan foydalanib, (5.6.10) Koshi masalasini hosil qilamiz. U holda (5.6.1) differensial tenglama $y = \varphi(t)$ yechimning

$$\lim_{t \rightarrow +\infty} (y_\delta(t) - \varphi(t)) = 0$$

asimptotik turg'unligi $z_\delta(t)$ holida quyidagi

$$\lim_{t \rightarrow +\infty} z_\delta(t) = 0$$

ko'rinishni oladi. Bu esa (5.6.7) bir jinsli tenglama $z(t) \equiv 0$ yechimini asimptotik turg'unligini bildiradi.

5.6.1-misol. Ushbu

$$y'(t) = ay(t), a \in R$$

tenglama $y(t) = 0$ yechimini turg'unlikka tekshiring.

Yechish. Berilgan differensial tenglamaning umumiy yechimi

$$y(t) = ce^{at}$$

ko'rinishda bo'ladi. Bu yerda $t \rightarrow +\infty$ da quyidagi uchta hol bo'lishi mumkin:

$$\begin{cases} a < 0, & y(t) \rightarrow 0, \\ a = 0, & |y(t)| \equiv const, \\ a > 0, & |y(t)| \rightarrow +\infty. \end{cases}$$

Ko'rinib turibdiki, $y(t) \equiv 0$ yechim $a < 0$ holida Lyapunov ma'nosida turg'un, bundan tashqari u asimptotik turg'un ham bo'ladi. Agar $a = 0$ bo'lsa, $y(t) \equiv 0$ yechim Lyapunov ma'nosida turg'un bo'ladi. Ammo $a > 0$ holida $y(t) \equiv 0$ yechim noturg'un bo'ladi.

5.6.2-misol. Quyidagi

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} y' + a_n y = 0, \quad a \in R, \quad a_0 \neq 0, \quad t > t_0 \quad (5.6.11)$$

o'zgarmas koeffitsiyentli bir jinsli tenglama $y(t) \equiv 0$ yechimini turg'unlikka tekshiring.

Yechish. Aytaylik, ushbu

$$a_0 \lambda^n + a_1 \lambda^{n-1} + \dots + a_{n-1} \lambda + a_n = 0 \quad (5.6.12)$$

xarakteristik tenglamaning k_1, k_2, \dots, k_n karrali har xil ildizlarini $\lambda_1, \lambda_2, \dots, \lambda_m \in C$ orqali belgilasak. U holda (5.6.11) differensial tenglamaning umumiy

yechimi quyidagi

$$e^{\lambda_1 t}, te^{\lambda_1 t}, \dots, t^{k_1-1} e^{\lambda_1 t};$$

$$e^{\lambda_2 t}, te^{\lambda_2 t}, \dots, t^{k_2-1} e^{\lambda_2 t};$$

$$\dots\dots\dots;$$

$$e^{\lambda_m t}, te^{\lambda_m t}, \dots, t^{k_m-1} e^{\lambda_m t};$$

funksiyalarning chiziqli kombinatsiyalaridan iborat bo'ladi. Bundan ko'rinib turibdiki, (5.6.11) tenglama $y(t) \equiv 0$ yechimi asimptotik turg'un bo'lishi uchun

$$\operatorname{Re}\{\lambda_j\} < 0, \forall j = \overline{1, n}$$

shartning bajarilishi zarur va yetarli. Agar (5.6.12) xarakteristik tenglamaning ildizlari orasida kamida bittasi, ya'ni

$$\exists j_0 \in N; \operatorname{Re}\{\lambda_{j_0}\} > 0$$

munosabatni qanoatlantirsa, u holda (5.6.11) differensial tenglamaning $y(t) \equiv 0$ yechimi noturg'un bo'ladi.

7-§. Ko'phadlarni turg'unlikka tekshirish

5.7.1-ta'rif. Agar haqiqiy koeffitsiyentli

$$P(\lambda) = a_0 \lambda^n + a_1 \lambda^{n-1} + \dots + a_{n-1} \lambda + a_n, a_0 > 0, a_j \in R, j = \overline{1, n} \quad (5.7.1)$$

ko'phadning barcha $\lambda = \lambda_j, P(\lambda_j) = 0, j = \overline{1, n}$ ildizlari ushbu

$$\operatorname{Re}\{\lambda_j\} < 0, j = \overline{1, n}$$

tengsizlikni qanoatlantirsa, unga turg'un ko'phad deyiladi.

Avvalo ushbu

$$P_1(\lambda) = a_0 \lambda + a_1, a_0 > 0, a_1 \in R \quad (5.7.2)$$

birinchi darajali ko'phadni ko'rib chiqamiz. Bu holda $P_1(\lambda) = 0$ tenglamaning ildizi

$$\lambda_1 = -\frac{a_1}{a_0}$$

ko'rinishda bo'ladi. Ko'rinib turibdiki, $\lambda_1 < 0$ bo'lishi uchun $a_1 > 0$ bo'lishi zarur va yetarli. Chunki $a_0 > 0$. Bundan kelib chiqadiki, birinchi darajali ko'phad turg'un bo'lishi uchun, uning barcha koeffitsientlari musbat, ya'ni $a_0 > 0, a_1 > 0$ bo'lishi zarur va yetarli.

Endi ikkinchi darajali

$$P_2(\lambda) = a_0 \lambda^2 + a_1 \lambda + a_2, a_0 > 0, a_1, a_2 \in R \quad (5.7.3)$$

ko'phadni qaraylik. Bu holda $P_2(\lambda) = 0$ tenglamaning ildizlari ushbu

$$\lambda_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0 a_2}}{2a_0}$$

formuladan topiladi. Bunda

1) agar $D = a_1^2 - 4a_0a_2 \leq 0$ bo'lsa, u holda

$$\operatorname{Re}\{\lambda_{1,2}\} = -\frac{a_1}{2a_0} \quad (5.7.4)$$

o'rinli.

2) Agar $D = a_1^2 - 4a_0a_2 > 0$ bo'lsa, u holda

$$\operatorname{Re}\{\lambda_{1,2}\} = \lambda_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0a_2}}{2a_0} \quad (5.7.5)$$

o'rinli. Ushbu $\operatorname{Re}\{\lambda_{1,2}\} < 0$ tengsizlik bajarilishi uchun quyidagi

$$\begin{cases} a_1 > 0, & \text{agar } a_1^2 - 4a_0a_2 \leq 0, \\ -a_1 + \sqrt{a_1^2 - 4a_0a_2} < 0, & \text{agar } a_1^2 - 4a_0a_2 > 0 \end{cases}$$

munosabatning o'rinli bo'lishi lozim. Bu munosabatning birinchisidan, ya'ni ushbu

$$a_1 > 0, \quad a_0a_2 \geq \frac{a_1^2}{4}$$

tengsizliklardan $a_2 > 0$ kelib chiqadi. Yuqoridagi munosabatning ikkinchisidan, ya'ni

$$\sqrt{a_1^2 - 4a_0a_2} < a_1$$

tengsizlikdan $a_1 > 0$, $a_1^2 - 4a_0a_2 < a_1^2$ baholar, bulardan esa $a_2 > 0$ ekani kelib chiqadi.

Shunday qilib, ikkinchi darajali $P_2(\lambda) = a_0\lambda^2 + a_1\lambda + a_2$, $a_0 > 0$, $a_1, a_2 \in R$ ko'phadning turg'un bo'lishi uchun uning barcha koeffitsiyentlarining musbat, ya'ni $a_0 > 0, a_1 > 0, a_2 > 0$ bo'lishi zarur va yetarli, ekan.

5.7.1-teorema. Birinchi va ikkinchi darajali $P_1(\lambda) = a_0\lambda + a_1$, $a_0 > 0, a_1 \in R$ $P_2(\lambda) = a_0\lambda^2 + a_1\lambda + a_2$, $a_0 > 0$, ko'phadlarning turg'un bo'lishi uchun, ularning barcha koeffitsiyentlarining musbat

$$a_0 > 0, \quad a_1 > 0, \quad a_2 > 0$$

bo'lishi zarur va yetarli.

5.7.2-teorema (Stodoliy). Ushbu

$$P(\lambda) = \lambda^n + a_1\lambda^{n-1} + \dots + a_{n-1}\lambda + a_n, \quad a_j \in R, \quad j = \overline{1, n}$$

ko'phad turg'un bo'lishi uchun uning barcha koeffitsiyentlari musbat $a_j > 0, j = \overline{1, n}$ bo'lishi zarur.

Isbot. Aytaylik berilgan $P(\lambda)$ ko'phad turg'un bo'lsin. U holda $a_j > 0, j = \overline{1, n}$ bo'lishini isbotlaymiz. Berilgan $P(\lambda)$ ko'phadning koeffitsiyentlari haqiqiy bo'lgani uchun uning ildizlari soni (karrali ildizlarning karrasi ham hisobga olinganda) n ta bo'ladi. Shu bilan birga

$P(\lambda)$ ko'phadning k ta ildizi kompleks bo'lsa, unda uning yana k ta ildizi mos ravishda qo'shma kompleks bo'ladi. Ularni $\alpha_j \pm i\beta_j$, $j=1,2,\dots,k$, λ_m , $m=1,2,\dots,n-2k$ deb belgilaymiz. Shuning uchun $\alpha_j < 0$, $j=1,2,\dots,k$, $\lambda_m > 0$, $m=1,2,\dots,n-2k$.

Endi $P(\lambda)$ ko'phadni quyidagicha yozamiz:

$$P(\lambda) = \prod_{j=1}^k \left\{ \left[\lambda - (\alpha_j + i\beta_j) \right] \cdot \left[\lambda - (\alpha_j - i\beta_j) \right] \right\} \prod_{m=1}^{n-2k} (\lambda - \lambda_m) = \\ = \prod_{j=1}^k (\lambda^2 + a_1^{(j)} \lambda + a_2^{(j)}) \prod_{m=1}^{n-2k} (\lambda + b^{(m)}).$$

Bunda $a_1^{(j)} = -2\alpha_j > 0$, $a_2^{(j)} = \alpha_j^2 + \beta_j^2 > 0$, $b^{(m)} = -\lambda_m > 0$. Demak, $P(\lambda)$ ko'phad koeffitsiyentlari musbat bo'lgan $\lambda^2 + a_1^{(j)} \lambda + a_2^{(j)}$ va $\lambda + b^{(m)}$ ko'rinishdagi ko'phadlarning ko'paytmasi shaklida yoziladi. Bunday ko'phadlarni o'zaro ko'paytirib chiqsak, natijada koeffitsiyentlari musbat bo'lgan ko'phad hosil bo'ladi. ■

5.7.1-izoh. Teskari tasdiq o'rinli emas, ya'ni barcha keffitsiyentlari musbat bo'lgan ko'phad turg'un bo'lavermaydi.

5.7.1-misol. Ushbu

$$\lambda^3 + \lambda^2 + 3\lambda + 10 = (\lambda + 2)(\lambda^2 - \lambda + 5)$$

uchinchi darajali ko'phadni qaraylik. Ko'rinib turibdiki bu ko'phadning ildizlari quyidagi

$$\lambda_1 = -2, \lambda_{2,3} = \frac{1 \pm i\sqrt{19}}{2}$$

sonlardan iborat. Bunda $Re\{\lambda_{2,3}\} = \frac{1}{2} > 0$. Demak, berilgan ko'phad noturg'un ekan.

5.7.1-lemma. Ushbu

$$P(\lambda) = \lambda^3 + a\lambda^2 + b\lambda + c, \quad a > 0, \quad b > 0, \quad c > 0$$

uchinchi darajali ko'phad sof mavhum ildizga ega bo'lishi uchun

$$a \cdot b = c$$

munosabatning bajarilishi zarur va yetarli.

Isbot. Zarurligi. Avvalo $\lambda = 0$ soni berilgan $P(\lambda)$ ko'phadning ildizi bo'la olmaydi. Chunki $P(0) = 0$ bo'lsa, $c = 0$ kelib chiqadi. Buning esa bo'lishi mumkin emas. Aytaylik, $\lambda_{1,2} = \pm i\mu$ ($\mu > 0$) soni $P(\lambda)$ ko'phadning ildizi bo'lsin. Bu holda uni

$$p(\lambda) = (\lambda^2 + \mu^2)(\lambda - \lambda_3) = \lambda^3 + (-\lambda_3)\lambda^2 + \mu^2\lambda + \mu^2(-\lambda_3)$$

ko‘rinishda yozish mumkin. Bunda $a = -\lambda_3, b = \mu^2, c = -\lambda_3\mu^2$. Ko‘rinib turibdiki, bu yerda ushbu

$$a \cdot b = c$$

munosabat bajariladi.

Yetarliligi. Aytaylik, ushbu

$$a \cdot b = c$$

tenglik o‘rinli bo‘lsin. U holda berilgan $P(\lambda)$ ko‘phadni ko‘paytuvchilarga ajratish mumkin bo‘ladi:

$$p(\lambda) = \lambda^3 + a\lambda^2 + b\lambda + c = \lambda^3 + a\lambda^2 + b\lambda + ab = (\lambda^2 + b)(\lambda + a).$$

Endi ushbu $p(\lambda) = 0, (\lambda^2 + b)(\lambda + a) = 0$ tenglamani qaraylik. Bundan

$$\lambda_{1,2} = \pm i\sqrt{b}; \lambda_3 = -a$$

ildizlarni topamiz. ■

5.7.3-teorema (Vishnegradskiy). Koeffitsiyentlari haqiqiy sonlardan iborat bo‘lgan

$$p(\lambda) = a_0\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3, \quad a_0 > 0 \quad (5.7.6)$$

uchinchi darajali ko‘phad turg‘un bo‘lishi uchun:

$$1) \quad a_0 > 0, a_1 > 0, a_2 > 0, a_3 > 0; \quad (5.7.7)$$

$$2) \quad a_1 \cdot a_2 > a_0 \cdot a_3 \quad (5.7.8)$$

shartlarning bajarilishi zarur va yetarli.

Isbot. Zarurligi. Avvalo

$$q(\lambda) = \lambda^3 + a\lambda^2 + b\lambda + c \quad (5.7.9)$$

ko‘phadni tuzib olamiz. Bunda

$$a = \frac{a_1}{a_0}; \quad b = \frac{a_2}{a_0}; \quad c = \frac{a_3}{a_0}.$$

Endi ushbu

$$a_1 = aa_0, \quad a_2 = ba_0, \quad a_3 = ca_0$$

tengliklardan foydalansak, (5.7.8) tengsizlik quyidagi ko‘rinishni oladi:

$$ab > c. \quad (5.7.10)$$

Aytaylik, (5.7.9) tenglik yordamida aniqlanadigan $q(\lambda)$ ko‘phad turg‘un bo‘lsin. U holda 5.7.2-teoremaga ko‘ra, $a > 0, b > 0, c > 0$ tengsizliklar bajariladi. Endi (5.7.10) tengsizlikning bajarilishini ko‘rsatamiz. Buning uchun teskarisini faraz qilamiz, ya’ni $q(\lambda)$ ko‘phad turg‘un bo‘lib, (5.7.10) tengsizlik bajarilmasin. U holda yoki $a \cdot b = c$, yoki $a \cdot b < c$ munosabatlar o‘rinli bo‘ladi. Berilgan $q(\lambda)$ ko‘phadni

$$p(\lambda) = (\lambda + a)(\lambda^2 + b) + c - ab \quad (5.7.11)$$

ko‘rinishda ifodalaymiz.

1-hol. Aytaylik, $ab = c$ bo‘lsin. U holda (5.7.11) tasvirdan

$$q(\lambda) = (\lambda + a)(\lambda^2 + b)$$

kelib chiqadi. Bu ko'phad $\lambda_{1,2} = \pm i\sqrt{b}$ ko'rinishdagi sof mavhum ildizga ega. Shuning uchun $q(\lambda)$ ko'phad noturg'un bo'ladi. Bu esa farazimizga zid.

2-hol. Aytaylik, $ab < c$ bo'lsin. Bu holda ham $q(\lambda)$ ko'phad noturg'un ekanini ko'rsatamiz. a va b larni ($a > 0, b > 0$) shunday uzluksiz o'zgartiramizki, birinchidan ular nolga intilsa, ikkinchidan $ab < c$ tengsizlik buzilmasin. Bunday o'zgartirishda $q(\lambda)$ ko'phadning ildizlari mavhum o'qning bir tomonidan ikkinchi tomoniga o'ta olmaydi, aks holda $ab < c$ tengsizlik buzilgan bo'ladi. Demak, $q(\lambda)$ ko'phadning turg'unligi yoki noturg'unligi o'zgarmaydi.

Agar $a = b = 0$ bo'lsa, u holda $q(\lambda) = \lambda^3 + c$ ko'rinishni oladi. Uning ildizlari $\lambda_1 = \sqrt[3]{-c} < 0$, $\lambda_{2,3} = \frac{\sqrt[3]{c}}{2} \pm i \frac{\sqrt{3}\sqrt[3]{c}}{2}$. Demak, $q(\lambda)$ ko'phad mavhum o'qdan o'ngda joylashgan ikkita $\frac{\sqrt[3]{c}}{2} \pm i \frac{\sqrt{3}\sqrt[3]{c}}{2}$ ildizga ega. Bu holda $q(\lambda)$ ko'phad noturg'un bo'ladi. Mazkur xossa a va b larning nolga yetarli yaqin qiymatlarida ham o'rinli. Chunki ildizlar ko'phad koeffitsiyentlarining uzluksiz funksiyasidir. Shunday qilib $ab < c$ tengsizlik bajarilganda $q(\lambda)$ ko'phad noturg'un bo'ladi.

Yetarliligi. Ushbu $ab > c$ tengsizlik bajarilsin, u holda $q(\lambda)$ ko'phadning turg'un ekanligini isbotlaymiz. Berilgan $ab > c$ tengsizlikda c ni shunday o'zgartiramizki u

1) nolga o'ngdan intilsin.

2) $ab > c$ tengsizlik buzilmasin.

Agar $c = 0$ bo'lsa $q(\lambda)$ ko'phad ushbu

$$q(\lambda) = \lambda(\lambda^2 + a\lambda + b)$$

ko'rinishni oladi. Bu ko'phad $\lambda_1 = 0$, $\lambda_{2,3} = -\frac{a}{2} \pm \sqrt{\frac{a^2}{4} - b}$ ko'rinishdagi ildizlarga ega. Bunda diskriminant ishorasiga bog'liq bo'lmagan holda, $\operatorname{Re}\{\lambda_{2,3}\} = -\frac{a}{2} < 0$ ekanligi ko'rinib turibdi. Haqiqatdan ham, agar

1) $D = a^2 - 4b \leq 0$ bo'lsa, u holda $\operatorname{Re}\lambda_{2,3} = -\frac{a}{2} < 0$ bo'ladi.

2) $D = a^2 - 4b > 0$ bo'lsa, u holda ushbu $a > \sqrt{a^2 - 4b}$ tengsizlikdan $\lambda_{2,3} < 0$ kelib chiqadi.

Agar c ning nolga yetarli yaqin musbat qiymatlarini olsak, $\lambda_{2,3}$ ildizlar mavhum o'qdan chapda, ya'ni $\operatorname{Re}\{\lambda_{2,3}\} < 0$ qoladi. Ammo nol ildiz mavhum o'qdan yoki chapga, yoki o'ngga yetarli kichik miqdorga siljiydi. Ikkinchi

tomondan ma'lumki, ko'phad ildizlarining ko'paytmasi teskari ishora bilan olingan ozod hadga teng. Shuning uchun qaralayotgan holda $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = -c < 0$, $\lambda_2 \cdot \lambda_3 = b > 0$ tengsizliklardan $\lambda_1 < 0$ ekani kelib chiqadi. Shunday qilib, $a > 0, b > 0, c > 0, ab > c$ tengsizliklar bajarilganda $q(\lambda)$ ko'phad turg'un bo'ladi. ■

5.7.1-natija. Vishnegradskiy teoremasidan quyidagi

$$a_0 y'''(t) + a_1 y''(t) + a_2 y'(t) + a_3 y(t) = 0, a_j > 0, j = 0, 3 \quad (5.7.12)$$

bir jinsli differensial tenglama $y(t) \equiv 0$ yechimining turg'unligi va noturg'unligi kelib chiqadi.

Agar $a_1 a_2 > a_0 a_3$ bo'lsa, u holda (5.7.12) differensial tenglamaning $y(t) \equiv 0$ yechimi asimptotik turg'un bo'ladi.

Agar $a_1 a_2 = a_0 a_3$ bo'lsa, u holda (5.7.13) differensial tenglamaning $y(t) \equiv 0$ yechimi turg'un bo'lib, asimptotik turg'un bo'lmaydi.

Agar $a_1 a_2 < a_0 a_3$ bo'lsa, u holda (5.7.12) differensial tenglamaning $y(t) \equiv 0$ yechimi noturg'un bo'ladi. ■

Eslatib o'tamizki, ushbu

$$A = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

matritsaning bosh minorlari deb quyidagi

$$M_1 = a_{11}, M_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \dots, M_j = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1j} \\ a_{21} & a_{22} & \dots & a_{2j} \\ \dots & \dots & \dots & \dots \\ a_{j1} & a_{j2} & \dots & a_{jj} \end{vmatrix}$$

determinantlarga aytiladi.

Quyidagi

$$P(\lambda) = a_0 \lambda^n + a_1 \lambda^{n-1} + \dots + a_{n-1} \lambda + a_n, a_0 > 0, a_j \in R, j = \overline{1, n} \quad (5.7.13)$$

ko'phadning Gurvis matritsasi deb ushbu

$$G = \begin{vmatrix} a_1 & a_0 & 0 & 0 & \dots & 0 & 0 \\ a_3 & a_2 & a_1 & a_0 & \dots & 0 & 0 \\ a_5 & a_4 & a_3 & a_2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & a_n \end{vmatrix}$$

matritsaga aytiladi. Bunda $a_j = 0, j > n$. Gurvis matritsasining bosh minorlari deb quyidagi determinantlarga aytiladi:

$$M_1 = a_1, M_2 = \begin{vmatrix} a_1 & a_2 \\ a_3 & a_1 \end{vmatrix}, M_3 = \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{vmatrix} \cdots, M_n = \begin{vmatrix} a_1 & a_0 & 0 & 0 & \dots & 0 \\ a_3 & a_2 & a_1 & a_0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{2n-1} & a_{2n-2} & \dots & a_n \end{vmatrix} = a_n M_{n-1}.$$

Bu yerda $a_j = 0, j > n$.

5.7.4-teorema (Raus-Gurvis belgisi). Haqiqiy koeffitsiyentli (5.7.13) ko'phad turg'un bo'lishi uchun quyidagi shartlarning bajarilishi zarur va yetarli:

1. Barcha koeffitsiyentlari musbat: $a_j > 0, j = 1, n$;
2. Gurvis matritsasining barcha bosh minorlari musbat: $M_j > 0, j = 1, n$.

5.7.5-teorema (Lenara-Shiparo belgisi). Haqiqiy koeffitsiyentli (5.7.13) ko'phad turg'un bo'lishi uchun quyidagi shartlarning bajarilishi zarur va yetarli:

1. $a_j > 0, j = 1, n$;
2. Gurvis matritsasining $j = n - 1, n - 3, n - 5, \dots$ nomerlariga mos keluvchi bosh minorlari musbat.

Bu ikki belgining ekvivalent ekanligini uchinchi darajali ko'phad misolida ko'rishimiz mumkin.

Haqiqatan ham, ushbu

$$P(\lambda) = a_0 \lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3. \quad a_0 > 0, \quad a_j \in R, \quad j = \overline{1, 3}$$

ko'phad uchun Gurvis matritsasi

$$G = \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ 0 & 0 & a_3 \end{vmatrix}$$

ko'rinishni oladi. Bu G matritsaning bosh minori quyidagi

$$M_1 = a_1, M_2 = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix}, M_3 = \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ 0 & 0 & a_3 \end{vmatrix}$$

determinantlardan iborat.

Berilgan uchinchi darajali $P(\lambda)$ ko'phad turg'un bo'lishi uchun, Linara-Shipara belgisiga ko'ra, ushbu

$$a_0, a_1, a_2, a_3 > 0, \quad M_2 > 0$$

shartlarning bajarilishi yetarli.

Endi $M_2 > 0$ tengsizlikdan $M_3 > 0$ bo'lishi kelib chiqishini ko'rsatamiz. Buning uchun M_3 determinantning oxirgi satr elementlari bo'yicha yoyamiz:

$$M_3 = \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ 0 & 0 & a_3 \end{vmatrix} = a_3 \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix} = a_3 M_2.$$

Agar $M_2 > 0$ bo'lsa, u holda bu tenglikdan va $a_j > 0, j = \overline{1,3}$ zaruriy shartlar bajarilganda $M_3 > 0$ kelib chiqadi.

Raus-Gurvis belgisidan foydalanib, o'zgarmas ko'effitsiyentli chiziqli bir jinsli

$$\frac{dx}{dt} = Ax, \quad A = \|a_{ij}\|, \quad a_{ij} = \text{const}, \quad i, j = \overline{1, n} \quad (5.7.14)$$

differensial tenglamalar sistemasi $x(t) = (x_1(t), \dots, x_n(t)) \equiv 0$ nol yechimini asimptotik turg'unlik shartini A matritsaning elementlari orqali ifodalash mumkin. Agar A matritsaning xarakteristik tenglamasini ushbu

$$\det\|\lambda E - A\| = \lambda^n - \sigma\lambda^{n-1} + s_2\lambda^{n-2} - \dots + (-1)^{n-1} s_{n-1}\lambda + (-1)^n \Delta$$

ko'rinishda yozib olsak. Bu yerda $\sigma = spA$, $\Delta = \det A$, $s_n = \sum_{k=2}^{n-1} M_k$. U holda,

xususan (5.7.14) sistemada $n=3$ bo'lsa $x(t) = (x_1(t), x_2(t), x_3(t)) \equiv 0$ yechimning asimptotik turg'un bo'lishi uchun

$$\sigma < 0, \quad \sigma s_2 < 0, \quad \Delta < 0 \quad (5.7.15)$$

shartlarning bajarilishi yetarli.

5.7.1-misol. Ushbu

$$\begin{cases} \dot{x}_1(t) = x_3, \\ \dot{x}_2(t) = -x_2 + x_3, \\ \dot{x}_3(t) = -x_1 + x_2 + px_3, \quad p < \frac{1-\sqrt{5}}{2} \end{cases}$$

differensial tenglamalar sistemasi $x_1(t) = 0, x_2(t) = 0, x_3(t) = 0$ yechimini asimptotik turg'unlikka tekshiring.

Yechish.

$$A = \begin{vmatrix} 0 & 0 & 1 \\ 0 & -1 & 1 \\ -1 & 1 & p \end{vmatrix},$$

$$\sigma = SpA = 0 - 1 + p = p - 1, \quad \Delta = \det A = -1,$$

$$s_2 = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ -1 & p \end{vmatrix} + \begin{vmatrix} -1 & 1 \\ 1 & p \end{vmatrix} = -p.$$

Berilgan p soni $p < \frac{1-\sqrt{5}}{2}$ tengsizlikni qanoatlantirsa, u holda

$$\sigma = p - 1 < \frac{1 - \sqrt{5}}{2} - 1 = -\left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right) < 0,$$

$$\sigma S_2 \leq (p - 1)(-p) = p(1 - p) < 0, \text{ chunki } p < 0, 1 - p > 0,$$

$$\Delta = -1 < 0$$

va (5.7.15) shartlar bajariladi. Demak, berilgan sistemaning $x(t) \equiv 0$ yechimi

$p < \frac{1 - \sqrt{5}}{2}$ tengsizligini qanoatlantiruvchi barcha p larda asimptotik turg'un bo'ladi.

Mustaqil yechishga doir misollar

Quyidagi differensial tenglamalar sistemasining muvozanat nuqtasini turg'unlikka tekshiring:

$$1. \begin{cases} \dot{x}_1(t) = \ln(x_2 + e^{2x_1}), \\ \dot{x}_2(t) = x_2 - 1 + \sqrt{1 - 3x_1}. \end{cases}$$

$$2. \begin{cases} \dot{x}_1(t) = -2x_1 + x_2 - x_1^2 + x_2^2, \\ \dot{x}_2(t) = -\operatorname{tg}x_2 + e^{x_1} - 1. \end{cases}$$

$$3. \begin{cases} \dot{x}_1(t) = x_2 - x_1 + 3x_1^2x_2 - \sin^2(x_1 + x_2), \\ \dot{x}_2(t) = 2x_2 - 3x_2 + \ln(1 + x_1^2x_2) - 6x_1^4 + x_2^3. \end{cases}$$

$$4. \begin{cases} \dot{x}_1(t) = -x_1 - \frac{1}{3}x_1^3 - x_1 \sin x_2, \\ \dot{x}_2(t) = -x_2 - \frac{1}{3}x_2^3. \end{cases}$$

$$5. \begin{cases} \dot{x}_1(t) = x_1x_2^2 - \frac{1}{2}x_1^3, \\ \dot{x}_2(t) = -\frac{1}{2}x_2^3 + \frac{1}{5}x_2x_1^2. \end{cases}$$

$$6. \begin{cases} \dot{x}_1(t) = -6x_1^2x_2, \\ \dot{x}_2(t) = -3x_2^2 + 6x_1. \end{cases}$$

VI BOB. AVTONOM (MUXTOR) SISTEMALAR VA UNING TRAYEKTORIYALARI

1-§. Muxtor sistemalar

Muxtor sistemalar differensial tenglamalar sistemasining muhim xususiy holdir. Ko'pchilik amaliy masalalarni yechish muxtor sistemalarni o'rganishga olib keladi.

6.1.1-ta'rif. Agar oddiy differensial tenglamalar sistemasiga erkli o'zgaruvchi oshkor ravishda kirmasa, bunga sistema muxtor sistema deyiladi.

Muxtor sistemaning quyidagi

$$\dot{x}(t) = f(x) \quad (6.1.1)$$

ko'rinishiga normal muxtor sistema deyiladi. Bu yerda $x(t) = (x_1(t), \dots, x_n(t))^T$ -noma'lum vektor-funksiya, $f(x) = (f_1(x), \dots, f_n(x))^T$ -berilgan vektor-funksiya.

Agar (6.1.1) sistemada erkli o'zgaruvchi t sifatida vaqt tushinilsa, unga dinamik sistema deyiladi.

Bundan keyin (6.1.1) differensial tenglamadagi $f(x)$ vektor-funksiya biror $D \subseteq R^n$ sohada aniqlangan, uzluksiz, differensiallanuvchi va $\frac{\partial f}{\partial x_i}, i = \overline{1, n}$

hosilalari chergaralangan deb qaraymiz. U holda D ga holatlar fazosi deyiladi. (6.1.1) sistemaning har bir $x_1 = x_1(t), x_2 = x_2(t), \dots, x_n = x_n(t)$ yechimiga n o'lchamli $D \subseteq R^n$ holatlar fazosida $x = (x_1, \dots, x_n)$ nuqtaning harakati mos keladi. Harakat davomida bu nuqta o'sha fazoda biror chiziqni (yoki, agar $x_i(t) = const, i = \overline{1, n}$ bo'lsa nuqtani) chizadi. Shu chiziqqa (yoki nuqtaga) x nuqtaning harakat trayektoriyasi (yoki holat trayektoriyasi) deyiladi. Ushbu $x = a$ nuqta trayektoriya bo'lishi uchun $f(a) = 0$ bo'lishi zarur va yetarli. Bunday nuqtaga maxsus nuqta yoki muvozanat nuqta deyiladi.

6.1.1-misol. Ushbu

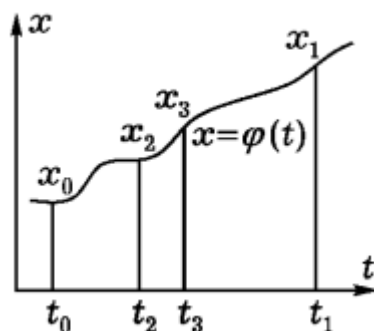
$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -x_1 \end{cases}$$

muxtor sistema

$$x_1 = c_1 \sin(t + \theta), x_2 = c_1 \cos(t + \theta) \quad (6.1.2)$$

ko'rinishidagi yechimga ega. Uch o'lchamli t, x_1, x_2 fazoda (6.1.2) tenglamalar vint chiziqni ifodalaydi. Holatlar fazosida (bu yerda ox_1x_2 tekislik) esa $x_1^2 + x_2^2 = c_1^2$ aylanalarni ifodalaydi. $x_1(t) = 0, x_2(t) = 0$ (maxsus nuqta) nuqta ham trayektoriya bo'ladi.

Muxtor sistemalarda $x = \varphi(t)$ (yechim) nuqtaning harakati to'g'risida to'liq ma'lumotga ega bo'lish uchun trayektoriyada t ning oshishiga mos harakat yo'nalishini ham berish lozim (1-chizma).



1-chizma

Agar $x = x(t)$ sistemaning trayektoriyasi bo'lsa, u holda $x(t) \neq 0$ bo'lib, o'zining har bir nuqtasida u $f(x) = (f_1(x), f_2(x), \dots, f_n(x))^T$ vektorga urinadi. Chunki $\dot{x}(t) = (\dot{x}_1(t), \dot{x}_2(t), \dots, \dot{x}_n(t))^T$ vektor, parametrik $x_1 = x_1(t)$, $x_2 = x_2(t)$, ..., $x_n = x_n(t)$ tenglamasi bilan berilgan chiziqqa urinadi va $\dot{x}(t) = f(x(t))$ tenglik o'rinli bo'ladi. Boshqacha aytganda, D to'plamning $x^0 \in D$ nuqtasiga shu nuqtadan chiqarilgan $f(x^0) = (f_1(x^0), f_2(x^0), \dots, f_n(x^0))^T$ vektorni mos qo'yamiz. Demak, (6.1.1) muxtor sistemaga D da aniqlangan vektor maydon mos keladi. $x^0 = (x_1^0, x_2^0, \dots, x_n^0) \in D$ bo'lsin. Mavjudlik va yagonalik teoremasiga ko'ra, (6.1.1) sistemaning $\varphi(t_0) = x^0$ boshlag'ich shartni qanoatlantiruvchi $x = \varphi(t)$ yechimi mavjud va yagona. Bu yechimga $t = t_0$ da trayektoriyasi $x^0 = (x_1^0, \dots, x_n^0)$ nuqtadan o'tuvchi nuqtaning harakati mos keladi. Harakat davomida $x = \varphi(t)$ yechimni belgilaydigan nuqtaning t_0 momentdagi tezligi $f(x^0)$ vektor bilan ifodalanadi, ya'ni

$$\varphi'(t)|_{t=t_0} = f(x^0).$$

Umuman olganda holatlar fazosini quyidagicha ta'riflash mumkin.

6.1.2-ta'rif. (6.1.1) muxtor sistemaning holatlar fazosi deb shunday n o'lchamli fazoga aytiladiki, unda shu sistemaning yechimlari trayektoriyalar bilan, sistemaning o'zi esa vektor maydon bilan tavsiflanadi. Bunda trayektoriyalar holat trayektoriyalari, vektorlar esa holat tezliklari deb ataladi.

6.1.1-teorema. Agar $x = \varphi(t)$, $t \in (\alpha, \beta)$ vektor-funksiya (6.1.1) sistemaning yechimi bo'lsa, u holda ixtiyoriy $c \in \mathbb{R}$ o'zgarmas son uchun $x = \psi(t) = \varphi(t + c)$, $t \in (\alpha - c, \beta - c)$ vektor-funksiya ham (6.1.1) sistemaning yechimi bo'ladi.

Isbot. Teorema shartiga ko'ra, $x = \varphi(t)$, $t \in (\alpha, \beta)$ vektor-funksiya (6.1.1) sistemaning yechimi bo'lgani uchun $\varphi'(t) \equiv f(\varphi(t))$ ayniyat o'rinli.

Bunda t ni $t + c$ ga almashtirsak,

$$\dot{\varphi}(t+c) \equiv f(\varphi(t+c))$$

hosil bo'ladi. Bundan

$$\dot{\psi}(t) = f(\psi(t)), \quad t \in (\alpha - c, \beta - c)$$

kelib chiqadi. ■

6.1.2-teorema. 1) Muxtor sistemaning ixtiyoriy ikki yechimiga mos keluvchi trayektoriyalari yoki kesishmaydi, yoki ustma-ust tushadi.

2) Aytaylik, $x = \varphi(t, \xi)$ funksiya (6.1.1) sistemaning $\varphi(0, \xi) = \xi$ boshlang'ich shartni qanoatlantiruvchi yechimi bo'lsin. U holda ixtiyoriy t_1 va t_2 lar uchun

$$\varphi(t_2, \varphi(t_1, x_0)) = \varphi(t_2 + t_1, x_0) \quad (6.1.2')$$

tenglik o'rinli bo'ladi. Bu tenglikka muxtor sistema yechimining gruppaviy xossasi deyiladi.

Isbot. 1) Aytaylik, $x(t)$ va $y(t)$ trayektoriyalar umumiy b nuqtaga ega bo'lsin. U holda shunday t_1 va t_2 topiladiki, natijada ushbu $b = x(t_1) = y(t_2)$ munosabatlar o'rinli bo'ladi. Ko'rinib turibdiki, $z(t) = y(t + t_2 - t_1)$ funksiya ham (6.1.1) sistemaning yechimi bo'ladi va $z(t_1) = y(t_2) = x(t_1)$ tengliklar bajariladi. Yagonalik teoremasiga ko'ra, $z(t) = x(t)$ o'rinli, ya'ni $y(t + t_2 - t_1) \equiv x(t)$. Bundan $x(t)$ va $y(t)$ yechimlar bir xil trayektoriyaga ega ekanligi kelib chiqadi. ■

Avtonom bo'lmagan sistemalar uchun (6.1.2') xossa bajarilmaydi.

6.1.1-misol. Avtonom bo'lmagan ushbu

$$\dot{x} = \cos t$$

tenglama $x(t) = \sin t$ ko'rinishdagi yechimga ega. Ammo $x(t) = \sin(t + C)$ ko'rinishdagi funksiya faqat $C = 2k\pi, k = 0, \pm 1, \dots$ bo'lganda uning yechimi bo'ladi. Berilgan tenglamaning $x(t) = \sin t$ va $x(t) = 1 + \sin t$ ko'rinishdagi har xil yechimlarini qaraylik. Ularning trayektoriyalari R^1 da joylashgan bo'lib, mos ravishda $-1 \leq x \leq 1$ va $0 \leq x \leq 2$ kesmalardan iborat bo'ladi. Bu trayektoriyalar har xil bo'lgani bilan ular kesishadi.

6.1.1-natija. Muxtor sistemasining yechimi vaqtning chekli qiymatida maxsus nuqtaga erisha olmaydi.

Isbot. Aytaylik, a – maxsus nuqta, ya'ni $\tilde{x}(t) \equiv a$ yechim bo'lsin. Agar $x(t)$ va $\tilde{x}(t)$ yechimlarning trayektoriyalari ustma-ust tushmasa, u holda ular umumiy nuqtaga ega bo'lmaydi. Shuning uchun $x(t) \neq a, \forall t$ larda o'rinli. $x(t)$ yechim faqat $t \rightarrow +\infty$ yoki $t \rightarrow -\infty$ bo'lganda maxsus nuqtaga yaqinlashishi mumkin. ■

6.1.3-teorema. Agar $x(t) \neq const$ yechim uchun $x(t_1) = x(t_2), t_2 > t_1$ munosabat bajarilsa, u holda bu yechim eng kichik musbat davrga ega

bo'lgan davriy vektor-funksiya bo'ladi, uning trayektoriyasi esa o'z-o'zini kesmaydigan yopiq chiziqdan iborat bo'ladi.

Isbot. Ushbu $y(t) = x(t + t_2 - t_1)$ funksiya 6.1.1-teoremaga ko'ra, yechim bo'ladi va $y(t_1) = x(t_2) = x(t_1)$ munosabatlar bajariladi. Yagonalik teoremasiga asosan $y(t) = x(t)$, ya'ni $x(t + d) = x(t)$, $d = t_2 - t_1 > 0$ davrga ega bo'lgan funksiyadan iborat bo'ladi. Boshqa davrlari ham bo'lishi mumkin. Teorema shartiga ko'ra, $x(t) \neq const$ bo'lgani uchun $\exists t^*$ topilib, $x(t^*) \neq x(t_1)$ o'rinli bo'ladi, ya'ni $|x(t^*) - x(t_1)| = r > 0$. $x(t)$ vektor-funksiyaning uzluksizligidan $r > 0$ uchun shunday $\exists h > 0$ soni topilib, $|t - t_1| < h$, ya'ni $t_1 - h < t < t_1 + h$ tengsizlik bajarilganda $|x(t) - x(t_1)| < r$ bajariladi. Bu esa $x(t) \neq x(t^*)$ munosabat $\forall t \in (t_1 - h, t_1 + h)$ larda bajarilishini bildiradi. Ammo vaqtning davrga teng qiymatida $x(t)$ yechim, ya'ni x nuqta trayektoriyaning barcha nuqtalarini bosib o'tishi lozim. Shuning uchun ixtiyoriy musbat davr uzunligi $2h$ dan oshmaydi va ularning quyi chegarasi $p \geq 2h$. Agar p soni davr bo'lmasa, u holda $p_i \rightarrow p + 0$ davrlar ketma-ketligi topilib $x(t + p_i) = x(t)$. Bunda $p_i \rightarrow p$ da limitga o'tib $x(t + p) = x(t)$ tenglikni olamiz, ya'ni p davr bo'ladi. $x(t)$ ($0 \leq t \leq p$) trayektoriya yopiq chiziqdan iborat bo'ladi. Chunki $x(0) = x(p)$. Agar u o'zini o'zi kessa, u holda $x(t_1) = x(t_2)$, $t_1, t_2 \in [0, p]$, $|t_1 - t_2| < p$ o'rinli. Yuqorida isbotlangan mulohazalarga asosan $x(t)$ yechim uchun $d = |t_2 - t_1| < p$ soni davr bo'ladi. Bu esa p ning eng kichik musbat davr ekanligiga zid keladi. ■

6.1.4-teorema. Muxtor sistemaning har bir trayektoriyasi quyidagi turlardan bittasiga tegishli bo'lishi mumkin:

- 1) muvozanat nuqta (nuqta);
- 2) o'z-o'zini kesmaydigan yopiq egri chiziq, ya'ni eng kichik musbat davrli yechimga mos keluvchi trayektoriya;
- 3) o'z-o'zini kesmaydigan yopiq bo'lmagan (ochiq) egri chiziq, ya'ni davriy bo'lmagan yechimga mos keluvchi trayektoriya.

Isbot. Agar muxtor sistemaning yechimi $x(t) = const$ bo'lsa, u holda trayektoriya nuqtada bo'ladi. Agar $x(t_1) = x(t_2)$, $\forall t_1$ va $t_2 \neq t_1$ bo'lsa, u holda trayektoriya -yopiq bo'lmagan o'z-o'zini kesmaydigan egri chiziqdan iborat bo'ladi. Agar $x(t) \neq const$ va $x(t_1) = x(t_2)$, $\exists t_1$ va $t_2 \neq t_1$ bo'lsa, u holda trayektoriya o'z-o'zini kesmaydigan yopiq egri chiziqdan iborat bo'ladi (6.1.3-teoremaga asosan). ■

6.1.5-teorema. Aytaylik, $x = \varphi(t)$ vektor-funksiya (6.1.1) sistemaning $r_1 < t < r_2$ intervalda aniqlangan biror yechimi bo'lsin. Agar $\varphi(t_1) = \varphi(t_2)$, $t_1 \neq t_2$ va $t_1, t_2 \in (r_1, r_2)$ bo'lsa, u holda shu $x = \varphi(t)$ yechimni $-\infty < t < \infty$ intervalga

davom ettirish mumkin.

Isbot. 6.1.1-teoremaga ko‘ra, $\varphi(t_1) = \varphi(t_2)$ bo‘lgani uchun $x = \varphi(t+c)$, $c = t_1 - t_2$ funksiya ham yechim bo‘ladi va $\varphi(t) = \varphi(t+c)$, $r_1 < t < r_2$ ayniyat o‘rinli. Bu ayniyatdan $\varphi(t)$ funksiya $r_1 < t < r_2$ intervalda aniqlangani uchun $\varphi(t+c)$ funksiya $r_1 - |c| < t < r_2 + |c|$ intervalda aniqlangan bo‘ladi. Haqiqatan ham $r_1 < t+c < r_2$ tengsizlikdan $c > 0$ bo‘lganda $r_1 - c < t < r_2$ va demak, yechimni r_1 dan chapga c miqdorga davom ettirish mumkin, shunga o‘xshash, $c < 0$ bo‘lganda $r_1 < t < r_2 - c$, ya’ni yechimni r_2 dan o‘ngga $-c = |c|$ miqdorga davom ettirish mumkin bo‘ladi. Har ikki holni birlashtirib yechimni $r_1 - |c| < t < r_2 + |c|$ intervalga davom ettirish mumkinligini qayd qilamiz. Shu intervalda aniqlangan $\varphi^{(1)}(t)$ yechim uchun $\varphi^{(1)}(t) = \varphi^{(1)}(t+c)$ ayniyat o‘rinli. $\varphi^{(1)}(t+c) = \varphi_*^{(1)}(t)$ desak, u holda $\varphi_*^{(1)}(t) = \varphi^{(1)}(t_1+c) = \varphi(t_1) = \varphi(t_2)$, ya’ni $\varphi_*^{(1)}(t_1) = \varphi(t_2)$ bundan xuddi avvalgidek $\varphi_*^{(1)}(t+c) = \varphi_*^{(1)}(t)$ ekaniligi kelib chiqadi. $\varphi_*^{(1)}(t)$ funksiya $r_1 - |c| < t < r_2 + |c|$ intervalda aniqlangan bo‘lgani uchun oxirgi ayniyatdan foydalanib, mavjudlik intervalini yanada kengaytirish mumkin. Boshqacha aytganda, $r_1 - 2|c| < t < r_2 + 2|c|$ intervalda aniqlangan yechimni qurish mumkin. Bu yechimni $\varphi^{(2)}(t)$ deb belgilaymiz. Shunga o‘xshash, mavjudlik intervalini $r_1 - k|c| < t < r_2 + k|c|$ dan iborat bo‘lgan $\varphi^{(k)}(t)$ yechimni qurish mumkin.

Yuqoridagi tengsizlikda $k \rightarrow \infty$ da limitga o‘tsak, $-\infty < t < \infty$ interval hosil bo‘ladi (r_1 va r_2 lar qanday bo‘lishidan qat’iy nazar). Shu intervalda aniqlangan yechimni $\varphi^0(t)$ deymiz. Ammo isbot davomida muxtor sistemaning har qanday trayektoriyasi chekli vaqtda cheksizga ketib qolmasligidan foydalanildi. ■

6.1.6-teorema. Agar D sohada $f_i(x_1, x_2, \dots, x_n)$, $i = \overline{1, n}$ funksiyalar barcha argumetlari bo‘yicha xususiy hosilalari chegaralangan bo‘lsa, u holda (6.1.1) muxtor sistemaning hech qanday trayektoriyasi chekli vaqtda cheksizga ketib qolmaydi, ya’ni ushbu

$$\lim_{t \rightarrow \tau} |\varphi(t)| = \infty$$

munosabat bajarilmaydi. Bu yerda $|\varphi(t)| = \sqrt{\varphi_1^2(t) + \dots + \varphi_n^2(t)}$.

Isbot. Teorema shartiga ko‘ra,

$$\left| \frac{\partial f_i}{\partial x_j} \right| \leq M, \quad i = \overline{1, n}, \quad j = \overline{1, n}.$$

Endi, $x = 0$ nuqta atrofida $f_i(x_1, x_2, \dots, x_n)$ funksiya uchun Lagranj formulasini yozamiz:

$$f_i(x) = f(0) + \frac{\partial f_i(\theta_i x)}{\partial x_1} x_1 + \dots + \frac{\partial f_i(\theta_i x)}{\partial x_n} x_n, \quad i = \overline{1, n} \quad (6.1.3)$$

$$0 < \theta_i < 1, \theta_i x \in D, |f(0)| = C.$$

Avvalo $\left| \frac{\partial f_i(\theta_i x)}{\partial x_i} \right|$ ifodani baholaymiz:

$$\left| \frac{\partial f_i(\theta_i x)}{\partial x_i} \right| = \sqrt{\left(\frac{\partial f_1(\theta_i x)}{\partial x_i} \right)^2 + \dots + \left(\frac{\partial f_n(\theta_i x)}{\partial x_i} \right)^2} \leq \sqrt{n} M.$$

Bundan foydalanib, $f(x)$ vektor-funksiyaning modulini baholash mumkin. Haqiqatan ham, (6.1.3) formulaga ko'ra

$$\begin{aligned} |f_j(x)| &\leq C + \sqrt{n} M \sum_{i=1}^n |x_i| \leq C\sqrt{n} + \sqrt{n} M \sum_{i=1}^n |x_i| = \\ &= \sqrt{n} \left(C + M \sum_{i=1}^n |x_i| \right) \leq N\sqrt{n} \left(1 + \sum_{i=1}^n |x_i| \right) \end{aligned}$$

tengsizlikni olamiz. Bu yerda $N = \max(C; M)$. Bu tengsizlikdan foydalanib, quyidagi

$$|f(x)| = \sqrt{\sum_{j=1}^n f_j^2(x)} = \sqrt{n^2 N^2 \left(1 + \sum_{i=1}^n |x_i| \right)^2} = nN \left(1 + \sum_{i=1}^n |x_i| \right)$$

bahoni olamiz. Faraz qilaylik, $r_1 < x < r_2 + \sum_{n=1}^k |C_n|$ intervalda aniqlangan va

$t \rightarrow \tau = r_2 + \sum_{m=1}^k C_m$ da cheksizlikka intiluvchi $x = \varphi(t)$ yechim mavjud, ya'ni

$t \rightarrow \tau$ da $|\varphi(t)| \rightarrow \infty$ ($\tau = r_1 - \sum_{m=1}^k C_m$ bo'lganda ham shunga o'xshash

isbotlanadi). U holda shunday $\tau^* < \tau$ topiladiki, $\tau^* \leq t < \tau$ intervalda $|\varphi(t)| > 1$ bo'ladi. Shuning uchun $\tau^* \leq t < \tau$ intervalda quyidagi

$$|\dot{\varphi}(t)| \leq |\dot{\varphi}_1(t)| + \dots + |\dot{\varphi}_n(t)| \leq Nn\sqrt{n} \left(1 + \sum_{i=1}^n |\varphi_i(t)| \right) \leq n(n+1)N\sqrt{n} |\varphi(t)|$$

bahoni keltirib chiqaramiz. Bundan

$$\frac{d}{dt} (|\varphi(t)|) \leq \frac{|\dot{\varphi}(t)|}{|\varphi(t)|} \leq n(n+1)N\sqrt{n}, \quad \tau^* \leq t < \tau$$

differensial tengsizlik kelib chiqadi. Oxirgi tengsizlikni ikki tomonini τ^* dan t gacha integrallab, quyidagini

$$|\varphi(t)| \leq |\varphi(\tau^*)| e^{n(n+1)N\sqrt{n}(t-\tau^*)}, \tau^* < t < \tau$$

bahoni topamiz. Ammo $t \rightarrow \tau$ da ushbu

$$|\varphi(\tau)| \leq |\varphi(\tau^*)| e^{n(n+1)N\sqrt{n}(t-\tau^*)}$$

tengsizlik o'rinli bo'lib, uning o'ng tomonidagi ifoda musbat chekli sonidir. Bu esa farazimizga zid. Demak, chekli vaqtda $x = \varphi(t)$ trayektoriya cheksizga keta olmaydi. ■

6.1.1-izoh. Aslida $\varphi(t)$ yechim chekli vaqtda cheksizga intilmasligi uchun $\varphi(t_1) = \varphi(t_2), t_1 \neq t_2$ munosabatning bajarilishi ham yetarli shartdir.

Endi $n=2$ holda (6.1.1) muxtor sistema yopiq trayektoriyaga ega bo'lmasligining yetarli shartini bayon qilamiz.

6.1.7-teorema. Agar (6.1.1) sistema tekislikdagi $D \subset R^2$ sohada berilgan bo'lib, D vektor maydon holat tezliklari, $f(x)$ potentsialli bo'lsa, u holda (6.1.1) sistema D sohada yopiq trayektoriyaga ega bo'lmaydi.

Isbot. Teskarisini faraz qilaylik. Aytaylik (6.1.1) muxtor sistema D sohada γ yopiq trayektoriyaga ega bo'lsin. Holat tezliklar maydoni potentsiali bo'lgani uchun ushbu

$$\oint_{\gamma} (f(x), dx) = 0$$

tenglik o'rinli bo'ladi (γ da yo'nalish soat strelkasiga qarshi). Ikkinchi tomondan, $x \in \gamma$ bo'lsa, $(f(x), f(x)) > 0$ va (1) tenglamani $dx = f[\varphi(t)]dt$ ko'rinishda yozib olamiz. Bu yerda $x = \varphi(t)$ (6.1.1) sistemaning $T > 0$ davrli yechimi bo'lib, γ -yopiq trayektoriyani aniqlaydi. Shuning uchun

$$\oint_{\gamma} (f(x), dx) = \int_0^T (f[\varphi(t)], f[\varphi(t)]) dt > 0$$

ziddiyat kelib chiqadi. ■

6.1.3-ta'rif. Quyidagi

1) $x = g(y)$ akslantirish D sohani D sohaga o'zaro bir qiymatli akslantiradi;

2) Ushbu $x = g(y), y \in D$ va $y = g^{-1}(x), x \in D$ vektor-funksiya mos ravishda D va D sohalarda uzluksiz differensiallanuvchi;

3) Ushbu munosabat o'rinli

$$\frac{\partial(g_1, g_2, \dots, g_n)}{\partial(y_1, y_2, \dots, y_n)} \neq 0, \forall y \in D$$

Shartlarni qanoatlantiruvchi $x = g(y)$ akslantirishga D sohada silliq teskarilanuvchi akslantirish deyiladi.

6.1.8-teorema. Aytaylik (6.1.1) sistemadagi $f(x)$ vektor-funksiya $D \subset R^n$ sohada uzluksiz differensiallanuvchi bo‘lib, $a \in D$ nuqtada $f(a) \neq 0$ bo‘lsin. U holda a nuqtaning D_a atrofi va shunday teskarilanuvchi almashtirish mavjud bo‘lib, shu D_a atrofda (6.1.1) sistema

$$\dot{y}_i = 0, i = \overline{1, n-1}, \dot{y}_n = 1$$

ko‘rinishni oladi. Bundan tashqari (6.1.1) sistemaning trayektoriyalari D_a atrofda $y_i = c_i, i = \overline{1, n-1}, y_n = t + c_n$ tog‘ri chiziq kesmalariga o‘tadi. Bunda $c_j, j = \overline{1, n}$ o‘zgarmas sonlar.

Isbot. Ushbu $a \in D$ nuqtada $f(a) \neq 0$ bo‘lgani uchun, $a = (a_1, a_2, \dots, a_n)$ nuqtaning koordinatalardan $a_n \neq 0$ deb hisoblaymiz. Endi $\xi = (\xi_1, \xi_2, \dots, \xi_{n-1}, a_n)$ vektorni tuzib olamiz. Bundan keyin (6.1.1) sistemaga qo‘yilgan ushbu

$$x(0) = \xi$$

Koshi masalasini qaraymiz. Mavjudlik va yagonalik teoremasiga ko‘ra, shunday $\varepsilon_1 > 0$ soni topilib $|t| \leq \varepsilon_1$ sohada aniqlangan (6.1.1) sistemaning $x = \varphi(t, \xi)$ yagona yechimi mavjud bo‘ladi. Ko‘rinib turibdiki, bu yechim $\varphi(0, \xi) = \xi$ boshlang‘ich shartni qanoatlantiradi. Koshi masalasining yechimini boshlang‘ich shartlarga nisbatan silliqligi haqidagi teoreмага asosan $x = \varphi(t, \xi)$ yechim t, ξ o‘zgaruvchilar bo‘yicha $|t| < \varepsilon_1, |\xi - a| < \varepsilon_2, \varepsilon_1 > 0, \varepsilon_2 > 0$ sohada differensiallanuvchi vektor funksiya bo‘ladi. Endi $x = \varphi(t, \xi)$ tenglama $t = 0, \xi = a$ nuqtaning biror atrofida silliq teskarilanuvchi almashtirishni ifodalashini ko‘rsatamiz. Shu maqsadda $t = 0, \xi = a$ nuqtaning biror atrofida oshkormas funksiyalar sistemasi yechimining mavjudligi va yagonaligi haqidagi teoremani qo‘llash mumkinligini ko‘rsatamiz. Ushbu

$$\varphi_i(0, \xi) = \xi_i, i = \overline{1, n-1}, \varphi_n(0, \xi) = a_n$$

boshlang‘ich shartlardan

$$\left. \frac{\partial \varphi_i}{\partial \xi_j} \right|_{\substack{t=0 \\ \xi=a}} = \delta_{i,j}, i = \overline{1, n}, j = \overline{1, n-1}$$

kelib chiqadi. Bu yerda $\delta_{i,j}$ -Kroneker simvoli. $x = \varphi(t, \xi)$ vektor-funksiya (6.1.1) sistemaning yechimi bo‘lgani uchun

$$\frac{\partial \varphi_i(0, a)}{\partial t} = f_i(a), i = \overline{1, n}$$

munosabat o‘rinli. Bundan quyidagi yakobiyanning noldan farqli ekanligi kelib chiqadi, ya’ni

$$\frac{\partial(\varphi_1, \varphi_2, \dots, \varphi_n)}{\partial(\xi_1, \xi_2, \dots, \xi_{n-1}, t)} \Big|_{\substack{\xi=a \\ t=0}} = f_n(a) \neq 0.$$

Oshkormas funksiyalar sistemasi yechimning mavjudligi va yagonaligi haqidagi teorema asosan $a \in D$ nuqtaning D_a atrofida aniqlangan uzluksiz differensiallanuvchi

$$\xi_i = u_i(x), i = \overline{1, n-1}, t = v(x)$$

yagona almashtirish mavjudligi kelib chiqadi. O'z navbatida bu almashtirish $x = \varphi(t, \xi)$ uchun teskari akslantirish vazifasini o'taydi. Endi D_a atrofda yangi o'zgaruvchilarni kiritamiz:

$$y_i = u_i(x), i = \overline{1, n-1}, y_n = v(x).$$

Bu D_a atrofda silliq teskarilanuvchi almashtirish. Har bir tayinlangan ξ da bu almashtirish natijasida (6.1.1) sistemaning D_a dagi trayektoriyalari

$$y_i = \xi_i, i = \overline{1, n-1}, y_n = t$$

to'g'ri chiziq kesmalariga o'tadi. $\xi \in D_a$ larni o'zgartirish hisobiga y_n o'qqa parallel kesmalar oilasini olamiz. (6.1.1) sistema yuqoridagi almashtirish natijasida ushbu

$$\dot{y}_i = 0, i = \overline{1, n-1}, \dot{y}_n = 1$$

ko'rinishni oladi. ■

Mustaqil yechish uchun mashqlar [13], §13, №1-52.

2-§. Trayektoriyalarning limitik to'plami

6.2.1-tarif. Agar $T(x = \varphi(t), -\infty < t < \infty)$ trayektoriya yoki $T^+(x = \varphi(t), t^* < t < \infty)$ yarim trayektoriya uchun shunday $\exists t_i \rightarrow \infty, \{t_i\}$ ketma-ketlik topilib, $i \rightarrow \infty$ da $\varphi(t_i) \rightarrow p$ munosabat o'rinli bo'lsa, p nuqtaga ω -limit nuqta deyiladi. T trayektoriyaning barcha ω -limit nuqtalari to'plamini $\Omega(T)$ deb belgilaymiz. Xuddi shuningdek T trayektoriyaning $t_i \rightarrow -\infty$ da α -limit nuqtalasini ham aniqlash mumkin. T trayektoriyaning barcha α -limit nuqtalari to'plamini $A(T)$ deb belgilaymiz.

Masalan, $x = e^t (t \in R)$ trayektoriya uchun $\Omega(T) = \emptyset$ bo'sh to'plam. Ammo $T^+(x = e^{-t}, 0 \leq t < \infty)$ yarim trayektoriya uchun $\Omega(T^+) = \{0\}$. Ushbu $T(x_1 = e^t \cos t (1 + e^t)^{-1}, x_2 = e^t \sin t (1 + e^t)^{-1})$ trayektoriya uchun $A(T) = \{(0, 0)\}$ dan iborat. Shu trayektoriya uchun $\Omega(T) = \{(x_1, x_2) \in R^2 : x_1^2 + x_2^2 = 1\}$ aylanadan iborat.

6.2.1-teorema. Aytaylik ushbu

$$\dot{x}(t) = f(x) \tag{6.2.1}$$

muxtor sistemadagi $f(x)$ vektor-funksiyaning $f_i(x), i = \overline{1, n}$ koordinatalari $G \subset R^n$ sohada uzluksiz va uzluksiz $\frac{\partial f_i}{\partial x_j}, i, j = \overline{1, n}$ xususiy hosilalarga ega bo'lsin. Agar $T^+(x = \varphi(t) \in R^n, t_1 \leq t < \infty)$ yarim trayektoriya chegaralangan bo'lib, o'zining ε atrofi bilan G sohada yotsa, u holda 1) $\Omega(T^+) \neq \emptyset$ 2) $\Omega(T^+)$ -kompakt, 3) $\Omega(T^+)$ -bog'lamli to'plam bo'ladi.

Isbot. Ixtiyoriy $t_i \rightarrow \infty$ ketma-ketlik uchun $\{\varphi(t_i)\}_{i=1}^{\infty}$ ketma-ketlik chegaralangan. Shuning uchun uning limit nuqtasi mavjud va u $\Omega(T^+) \neq \emptyset$ bo'sh bo'lmagan to'plam bo'ladi. T^+ -yarim trayektoriyaning chegaralanganligidan $\Omega(T^+)$ -to'plamning chegaralanganligi kelib chiqadi. Endi $\Omega(T^+)$ ning yopiqqligini ko'rsatamiz. Agar $p_i \in \Omega(T^+), p_i \rightarrow p, (i = \overline{1, \infty}) i \rightarrow \infty$ bo'lsa, u holda $\eta = 2^{-i}$ uchun shunday $\exists p_i$ topilib $|p_i - p| < \eta$ o'rinli bo'ladi. Shu p_i lar uchun shunday $t_{i,j}, j = \overline{1, n}$ ketma-ketlik topilib $\varphi(t_{i,j}) \rightarrow p_i (j \rightarrow \infty)$ o'rinli. Shuning uchun shunday $\exists j = i(j) \in N$ nomer topilib, $t_{i,i(j)} > i$ tengsizlik bajarilganda $|\varphi(t_{i,i(j)}) - p_i| < \eta$ o'rinli. U holda $|\varphi(t_{i,i(j)}) - p| < 2\eta = 2^{1-i} t_{i,i(j)} > i \rightarrow \infty$. Shuning uchun $p \in \Omega(T^+)$ o'rinli bo'ladi. Demak $\Omega(T^+)$ chegaralangan va yopiq to'plam bo'lgani uchun u kompakt to'plamdir. Nihoyat $\Omega(T^+)$ to'plamning trayektoriyalardan tuzilganligini ko'rsatamiz, ya'ni har bir $a \in \Omega(T^+)$ nuqtadan $T_a(x = z(t), -\infty < t < \infty)$ trayektoriya o'tishini va u $\Omega(T^+)$ da joylashishini isbotlaymiz. $\Omega(T^+)$ to'plam ta'rifidan $\exists t_1, t_2, \dots, t_n, \dots \rightarrow \infty$ ketma-ketlik topilib, $\varphi(t_i) \rightarrow a (i \rightarrow \infty)$ o'rinli bo'lishi kelib chiqadi. Ushbu $\chi_i(t) \equiv \varphi(t_i + t), (i = 1, 2, 3, \dots)$ vektor-funksiya yechim bo'lib, $\chi_i(0) \rightarrow a (i \rightarrow \infty)$ o'rinli. Endi (6.2.1) sistemaning $z(0) = a$ boshlang'ich shartni qanoatlantiruvchi yechimini $z(t)$ deb, F orqali T^+ yarim trayektoriyaning $\frac{\varepsilon}{2}$ atrofini belgilaymiz. Ushbu $t = 0, x = a$ nuqta $D(-\infty < t < \infty, x \in F)$ yopiq chegaralanmagan to'plamga qarashli. Yechimni davom qildirish mumkinligi haqidagi teoremaga muvofiq $z(t)$ yechimni ikki tomonga yetarli katta $|t|$ lar uchun davom qildirish mumkin bo'ladi. Faraz qilaylik $z(t)$ ushbu $t = t^*, x = z(t^*) = b$ nuqtada D sohaning chegarasiga chiqsin. U holda $b \in \partial F$, ya'ni F to'plamning chegarasida yotadi. Bundan

$$\rho(b, T^+) = \frac{\varepsilon}{2} \quad (6.2.2)$$

ekanligi kelib chiqadi. Ushbu $\chi_i(0) \rightarrow a = z(0)$ munosabatdan

$$\chi_i(t^*) \rightarrow z(t^*) = b(i \rightarrow \infty) \quad (6.2.3)$$

kelib chiqadi. $\chi_i(t^*) = \varphi_i(t_i + t^*) \in T^+$, $t_i > t_1 - t^*$ munosabatni inobatga olsak, u holda (6.2.3) munosabat (6.2.2) ni inkor etadi. Shuning uchun $z(t)$ D sohaning chegarasiga chiqmaydi, ya'ni $(-\infty; \infty)$ intervalga davom etadi. U holda yuqoridagiday $\forall t$ uchun $\varphi(t_i + t) = \chi_i(t) \rightarrow z(t)(i \rightarrow \infty)$. Bu esa, $\forall t$ da $z(t) \in \Omega(T^+)$, ekanligini ko'rsatadi.

Endi $\Omega(T^+)$ ning bog'lamli to'plam bo'lishini isbot qilamiz. Faraz qilaylik, $\Omega(T^+)$ kompakt bo'lib, bog'lamli to'plam bo'lmasin. U holda uni ushbu $\Omega(T^+) = \Omega_1 \cup \Omega_2$ ko'rinishda ifodalash mumkin. Bunda $\Omega_1 \cap \Omega_2 = \emptyset$ bo'lib, $\Omega_1 \neq \emptyset$, $\Omega_2 \neq \emptyset$. Bundan tashqari $\Omega_i, i=1,2$ to'plamlarning har biri ikkinchisining limit nuqtasini o'zida saqlamaydi. Bu to'plamlar orasidagi masofani $\rho(\Omega_1; \Omega_2) = d$ orqali belgilaymiz. $\Omega(T^+)$ kompakt bo'lgani uchun $\Omega_i, i=1,2$ ham kompakt bo'ladi. Bundan $d > 0$ ekanligi kelib chiqadi.

Shunday $\exists \bar{t}_k \rightarrow \infty, \overline{\bar{t}_k} \rightarrow \infty$ ketma-ketliklar topiladiki, bular uchun $\rho(\varphi(\bar{t}_k), \Omega_1) < \frac{d}{2}$ va $\rho(\varphi(\overline{\bar{t}_k}), \Omega_2) \leq \frac{d}{2} \quad \forall k \in N$ munosabatlar bajariladi.

Umumiylikni buzmaganda holda $\bar{t}_1 < \bar{t}_2 < \dots < \overline{\bar{t}_k} < \overline{\bar{t}_k} < \dots$ deb hisoblaymiz. Endi $\bar{\Omega}_1$ va $\bar{\Omega}_2$ orqali mos ravishda Ω_1 va Ω_2 to'plamlarning $\frac{d}{2}$ atrofini

belgilaymiz. Aytaylik $\bar{\Omega}_1 \cup \bar{\Omega}_2 = \Omega_0$ bo'lsin. Bunda $\bar{\Omega}_1$ va $\bar{\Omega}_2$ ochiq to'plamlar. Ω_0 to'plam ochiq lekin, bog'lamli emas. Chunki, $\bar{\Omega}_1 \cap \bar{\Omega}_2 = \emptyset$. Bundan ko'rinadiki, $\bar{\Omega}_1$ va $\bar{\Omega}_2$ larning nuqtalarini Ω_0 da yotadigan chiziq (yo'l) bilan tutashtirib bo'lmaydi. Shunday qilib, har bir k uchun shunday $\exists t_k \in (\bar{t}_k, \overline{\bar{t}_k})$

topiladiki, uning uchun $\rho(\varphi(t_k), \Omega_1) \geq \frac{d}{2}$, $\rho(\varphi(t_k), \Omega_2) \geq \frac{d}{2}$ o'rinli bo'ladi.

Aniqroq aytadigan bo'lsak, $\{\varphi(t_k)\}$ to'plam $q^* \in \Omega(T)$ limit nuqtaga ega bo'ladi. Bunday bo'lishi mumkin emas, chunki $\rho(q^*, \Omega_1) \geq \frac{d}{2}$, $\rho(q^*, \Omega_2) \geq \frac{d}{2}$.

Bu qarama-qarshilik $\Omega(T^+)$ to'plamning bog'lamli ekanini bildiradi. ■

R^2 tekislikda berilgan muxtor sistema limitik to'plamining boshqa xossalari ham o'rganilgan.

6.2.2-teorema(Bendikson). Tekislikdagi chegaralangan o'zida maxsus nuqtani saqlamaydigan ω -limitik to'plam yopiq trayektoriyadan iborat bo'ladi.

Ushbu R^n , $n \geq 3$ fazoda ω -limitik to'plamning tuzilishi kam o'rganilgan.

3-§. Chiziqli o'zgaras koeffitsiyentli bir jinsli differensial tenglamalar sistemasi muvozanat (maxsus) nuqtasining klassifikatsiyasi

Ushbu

$$\begin{cases} \dot{x}_1 = a_{11}x_1 + a_{12}x_2, \\ \dot{x}_2 = a_{21}x_1 + a_{22}x_2 \end{cases} \quad (6.3.1)$$

o'zgaras koeffitsiyentli chiziqli bir jinsli differensial tenglamalar sistemasini qaraylik. Bu yerda $a_{ij} \in R$ haqiqiy sonlar $t \in R$ -erkli o'zgaruvchi, $x(t) = (x_1, x_2)^T$ -noma'lum vektor funksiya. Ko'rinib turibdiki, (6.3.1) -muxtor sistema. Berilgan differensial tenglamalar sistemasining koeffitsiyentlaridan

$$A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

matritsa tuzib olamiz.

6.3.1.-ta'rif. Agar A - xosmas matritsa bo'lsa, u holda (6.3.1) sodda sistema, aks holda murakkab sistema deyiladi.

Avvalo (6.3.1) sodda sistemani ushbu

$$x(t)|_{t=0} = 0. \quad (6.3.2)$$

boshlang'ich shartni qanoatlantiruvchi yechimi $x(t) = 0$, ya'ni $x_1(t) = 0$, $x_2(t) = 0$ ekanligi ravshan. Bundan tashqari (6.3.1) sodda muxtor sistemaning muvozanat (maxsus) nuqtasi ushbu $Ax = 0$, $x = (x_1, x_2)^T$ tenglamadan aniqlanar edi. A matritsa xosmas bo'lgani uchun, undan $x = 0$, ya'ni $(0, 0)$ kelib chiqadi. Bundan buyon $(0, 0)$ muvozanat nuqtani turg'unlikka tekshiramiz va sodda muxtor sistema trayektoriyalarini o'rganamiz.

Berilgan (6.3.1) sistemaning umumiy yechimini topish uchun A matritsaning xos qiymatlarini va xos vektorlarini topamiz. Buning uchun ushbu

$$Ah = \lambda h, 0 \neq h = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

tenglamani qaraymiz. Bu tenglamani koordinatalarda yozib,

$$\begin{cases} (a_{11} - \lambda)x_1 + a_{12}x_2 = 0 \\ a_{21}x_1 + (a_{22} - \lambda)x_2 = 0 \end{cases}$$

bir jinsli algebraik tenglamalar sistemasini hosil qilamiz. Ma'lumki, bir jinsli algebraik tenglamalar sistemasi nolmas yechimga ega bo'lishi uchun

$$\det(A - \lambda I) = \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = 0$$

bo'lishi zarur va yetarli. Bundan ushbu

$$\lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} = 0,$$

ya'ni

$$\lambda^2 - (spA)\lambda + \det A = 0. \quad (6.3.3)$$

kvadrat tenglama kelib chiqadi. Ko'rinib turibdiki, $\lambda = 0$ soni bu kvadrat tenglamaning ildizi bo'lmaydi. Chunki $\det A \neq 0$. A matritsaning λ_1, λ_2 -xos qiymatlari (6.3.3) xarakteristik tenglamaning ildizlaridan iborat bo'lishi ma'lum. Quyidagi hollarni o'rganamiz.

1-hol. A matritsaning xos qiymatlari haqiqiy va $\lambda_1 \neq \lambda_2$ har xil bo'lsin. Bu xos qiymatlarga mos keluvchi xos vektorlarni $h_1 = (\alpha_1, \alpha_2)^T$, $h_2 = (\beta_1, \beta_2)^T$ deb belgilaylik. U holda

$$\begin{cases} (a_{11} - \lambda_1)\alpha_1 + a_{12}\alpha_2 = 0, \\ a_{21}\alpha_1 + (a_{22} - \lambda_1)\alpha_2 = 0 \end{cases}$$

va

$$\begin{cases} (a_{11} - \lambda_2)\beta_1 + a_{12}\beta_2 = 0, \\ a_{21}\beta_1 + (a_{22} - \lambda_2)\beta_2 = 0 \end{cases}$$

sistemalarga ega bo'lamiz. Berilgan (6.3.1) muxtor sistemaning umumiy yechimi ushbu

$$x(t) = C_1 e^{\lambda_1 t} h_1 + C_2 e^{\lambda_2 t} h_2 \quad (6.3.4)$$

ko'rinishda bo'lishini oldingi paragraflarda ko'rgan edik. h_1, h_2 -xos vektorlar R^2 fazoning bazis vektorlaridan iborat bo'lib, ular umuman olganda ortogonal emas. Agarda ξ_1 va ξ_2 orqali $x \in R^2$ nuqtaning h_1, h_2 bazisdagi koordinatalarini belgilasak, u holda ushbu $x(t) = \xi_1(t)h_1 + \xi_2(t)h_2$ yoyilmadan va (6.3.4) formuladan $x(t)$ -yechimning koordinatalari

$$\xi_1(t) = C_1 e^{\lambda_1 t}, \quad \xi_2(t) = C_2 e^{\lambda_2 t}$$

ko'rinishni oladi.

a) Aytaylik, $\lambda_1 < 0, \lambda_2 < 0$ va $|\lambda_1| < |\lambda_2|$ bo'lsin. U holda $C_1 = 0, C_2 = 0, x(t) = 0, (0,0)$ muvozanat nuqtani ifodalaydi. (6.3.4) umumiy yechimni tarkibida $e^{-|\lambda_1|t}, e^{-|\lambda_2|t}$ hadlar qatnashganligi uchun, ushbu

$$\lim_{t \rightarrow +\infty} x(t) = 0$$

munosabat o'rinli bo'ladi. Shuning uchun $(0,0)$ muvozanat (maxsus) nuqta asimptotik turg'un bo'ladi. Bu holda $(0,0)$ -muvozanat (maxsus) nuqtaga turg'un tugun deyiladi.

Agar $C_1 > 0, C_2 = 0$ bo'lsa $\xi_1 > 0$ bo'lib, $t \rightarrow +\infty$ da $\xi_1(t) \rightarrow +0$ bo'ladi. Agar $C_1 = 0, C_2 > 0$ bo'lsa $\xi_2 > 0$ bo'lib, $t \rightarrow +\infty$ da $\xi_2(t) \rightarrow +0$ bo'ladi. Agar

$C_1 > 0, C_2 > 0$ bo'lsa, u holda $\xi_1 = C_1 e^{\lambda_1 t}$ dan $t = \frac{1}{\lambda_1} \ln \frac{\xi_1}{C_1}$ ni aniqlaymiz va

$$\xi_2 = C_2 e^{\frac{\lambda_2}{\lambda_1} \ln(\frac{\xi_1}{C_1})} = C_2 \left(\frac{\xi_1}{C_1}\right)^{\frac{\lambda_2}{\lambda_1}} = C_2 C_1^{-\frac{\lambda_2}{\lambda_1}} \cdot \xi_1^{\frac{\lambda_2}{\lambda_1}}$$

ko‘rinishda bo‘lishini topamiz. Bunda ushbu

$$C = C_2 C_1^{-\frac{\lambda_2}{\lambda_1}}, \alpha = \frac{\lambda_2}{\lambda_1} > 1$$

belgilashni kiritsak, quyidagi

$$\xi_2(t) = C \xi_1^\alpha$$

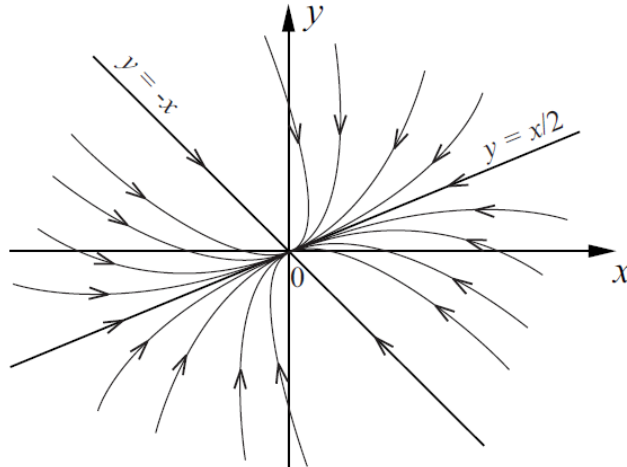
tenglamaga ega bo‘lamiz. Bu esa parabolani ifodalovchi egri chiziqlar oilasidir. Bunda quyidagi

$$\lim_{t \rightarrow +\infty} \frac{d\xi_2}{d\xi_1} = C\alpha \lim_{t \rightarrow +\infty} \xi_1^{\alpha-1} = 0$$

munosabat o‘rinli. Demak, trayektoriyalar parabola shoxchalaridan iborat bo‘lib, ξ_1 o‘qqa koordinata boshida urinadi. Parabolani ifodalovchi egri chiziqlarni ikki to‘g‘ri chiziq ajratib turadi. Ulardan biri $C_1 = 0$, ya’ni

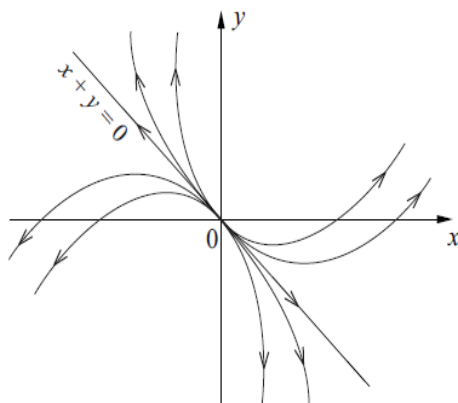
$$x = C_2 \beta_1 e^{\lambda_2 t}, y = C_2 \beta_2 e^{\lambda_2 t}, y = \frac{\beta_1}{\beta_2} x.$$

Ikkinchisi esa $y = \frac{\alpha_1}{\alpha_2} x$ to‘g‘ri chiziqdir. 1-chizmaga qarang



1-chizma

b) Aytaylik, $\lambda_1 > 0, \lambda_2 > 0$ va $\lambda_1 < \lambda_2$ bo‘lsin. Bu holda t ni $-t$ ga almashtirish natijasida avvalgi holga o‘tadi. Bu holda ham trayektoriya xuddi avvalgi holdagi kabi bo‘ladi, ammo trayektoriya bo‘yicha harakat qarama-qarshi tomonga yo‘nalgan bo‘ladi. Bu holda $(0,0)$ muvozanat nuqtadan uzoqlashish ham sodir bo‘ladi. Bunda $(0,0)$ muvozanat (maxsus) nuqta turg‘unmas tugun deyiladi.



2-chizma

v) Aytaylik, $\lambda_1 > 0, \lambda_2 < 0$ ($\lambda_1 < 0 < \lambda_2$) bo'lsin. U holda berilgan sodda muxtor sistemaning umumiy yechimi ushbu

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = C_1 e^{\lambda_1 t} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} + C_2 e^{\lambda_2 t} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$$

ko'rinishda bo'ladi. Bu yerda ham x nuqtaning $h_1 = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$ va $h_2 = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$

bazisdagi koordinatalarini mos ravishda ξ_1 va ξ_2 orqali belgilasak,

$$\xi_1(t) = C_1 e^{\lambda_1 t}, \quad \xi_2(t) = C_2 e^{\lambda_2 t}$$

munosabatlarga ega bo'lamiz.

Agar $C_1 = C_2 = 0$ bo'lsa, $x(t) = 0$, ya'ni $(0,0)$ muvozanat nuqtaga ega bo'lamiz. Agar $C_1 > 0, C_2 = 0$ bo'lsa, $\xi_1 > 0$ bo'lib, $t \rightarrow \infty$ da $\xi_1(t) \rightarrow +\infty$ bo'ladi. Agar $C_1 = 0, C_2 > 0$ bo'lsa, $\xi_2 > 0$ bo'lib, $t \rightarrow \infty$ da $\xi_2(t) \rightarrow +\infty$ bo'ladi. Agar $C_1 > 0, C_2 > 0$ bo'lsa, u holda a) holdagidek

$$\xi_2 = C \xi_1^\alpha, \quad \alpha = \frac{\lambda_2}{\lambda_1} < 0$$

munosabatga ega bo'lamiz. Bu esa giperbola tipidagi egri chiziqlarni ifodalaydi. Shuni alohida qayd qilish lozimki, bu egri chiziqlarni quyidagi to'g'ri chiziqlar ajratib turadi. Umumiy yechim tarkibidagi o'zgarmaslardan $C_1 = 0$ bo'lsa, u holda yechim

$$x_1 = C_2 e^{\lambda_2 t} \beta_1, \quad x_2 = C_2 e^{\lambda_2 t} \beta_2$$

ko'rinishni oladi. Bu yechimga mos keluvchi trayektoriya

$$y = \frac{\beta_2}{\beta_1} x$$

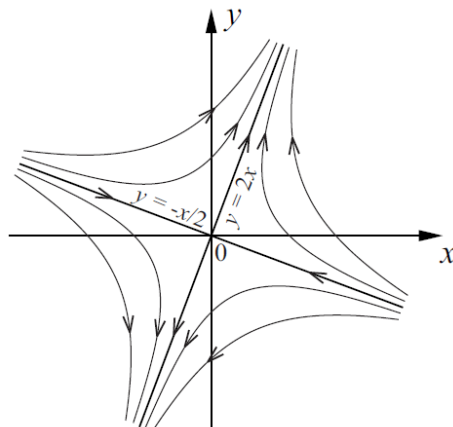
ko'rinishda bo'ladi. Biz qarayotgan holda $\lambda_2 < 0$ bo'lgani uchun to'g'ri chiziq bo'yicha harakat koordinata boshi tomon yo'nalgan bo'ladi. Xuddi shuningdek, $C_2 = 0$ desak

$$x_1 = C_1 \alpha_1 e^{\lambda_1 t}, \quad x_2 = C_1 \alpha_2 e^{\lambda_1 t}$$

yechimga ega bo'lamiz. Bu yechimga mos keluvchi trayektoriya

$$y = \frac{\alpha_2}{\alpha_1} x$$

to'g'ri chiziqdan iborat bo'ladi. Qaralayotgan holda $\lambda_1 > 0$ bo'lgani uchun to'g'ri chiziq bo'yicha harakat koordinata boshidan uzoqlashadi. Bu ikki to'g'ri chiziqqa separatsiya deyiladi. Qolgan trayektoriyalarning barchasi giperbola ko'rinishida bo'ladi. Bu holda $(0,0)$ muvozanat (maxsus) nuqtaga egar deyiladi.

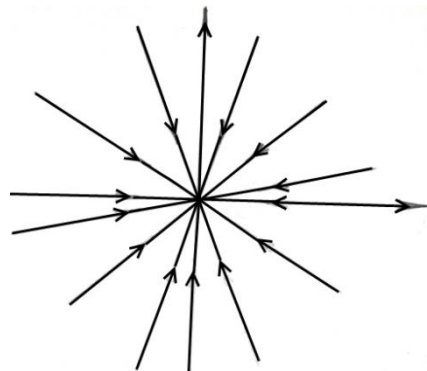


3-chizma

g) 1) Aytaylik, $\lambda_1 = \lambda_2 = \lambda$ bo'lib, R^2 tekislikda A matritsaning h_1, h_2 xos vektorlari bazisni tashkil qilsin. Bu holdda (6.3.1) muxtor sistemaning yechimi

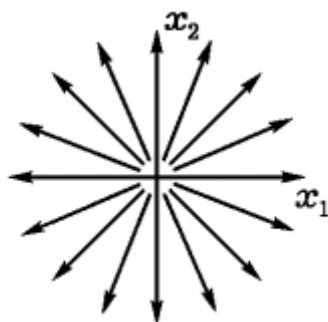
$$x(t) = e^{\lambda t} (C_1 h_1 + C_2 h_2)$$

ko'rinishda bo'ladi. Bunday ko'rinishdagi har bir yechim nurni ifodalaydi. Agar $\lambda < 0$ bo'lsa, bu nurlar bo'yicha harakat $t \rightarrow +\infty$ da nolga yaqinlashadi. Chunki, $t \rightarrow +\infty$ da $x(t) \rightarrow 0$ bo'ladi. Agar $\lambda > 0$ bo'lsa, bu nurlar bo'yicha harakat noldan uzoqlashadi. Chunki, $t \rightarrow +\infty$ da $x(t) \rightarrow +\infty$ bo'ladi. Ushbu $\lambda < 0$ holda $(0,0)$ muvozanat (maxsus) nuqta dikritik turg'un tugun deyiladi.



4-chizma

Ushbu $\lambda > 0$ holda $(0,0)$ muvozanat (maxsus) nuqta dikritik noturg'un tugun deyiladi. Chunki, bu holda trayektoriyalar xuddi oldingidek bo'ladi, ammo harakat yo'nalishi qarama-qarshi tomonga yo'nalgan bo'ladi.



5-chizma

2) Aytaylik, $\lambda_1 = \lambda_2 = \lambda$ bo'lib R^2 tekislikda h_1, h_2 vektorlar Jordan bazislarini tashkil qilsin. Bunda $Ah_1 = \lambda h_1$, h_1 - xos vektor, $Ah_2 = \lambda h_2 + h_1$, h_2 - yopishgan vektor. Bu holda (6.3.1) muxtor sistemaning umumiy yechimi

$$x(t) = C_1 e^{\lambda t} h_1 + C_2 e^{\lambda t} [t h_1 + h_2]$$

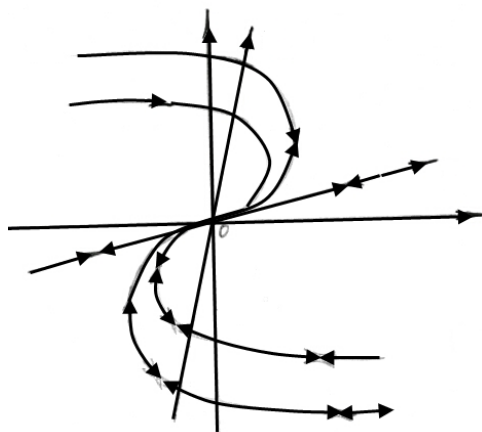
ko'rinishda bo'ladi. x nuqtaning h_1, h_2 bazisdagi koordinatalarni mos ravishda ξ_1, ξ_2 deb belgilasak,

$$\xi_1(t) = (C_1 + C_2 t) e^{\lambda t}, \quad \xi_2(t) = C_2 e^{\lambda t}$$

munosabatga ega bo'lamiz.

a) Agar $\lambda < 0$ bo'lib, $C_1 = C_2 = 0$ bo'lganda $x = 0, (0,0)$ -muvozanat (maxsus) nuqta hosil bo'ladi. $C_1 \neq 0, C_2 = 0$ bo'lganda $\xi_1 < 0$ va $\xi_2 > 0$ bo'lib, $t \rightarrow +\infty$ da harakat $x = 0$ ga yaqinlashadi. Umuman olganda yechimning umumiy ko'rinishidan qaralayotgan $\lambda < 0$ holida $t \rightarrow +\infty$ da $x(t) \rightarrow 0$ bo'ladi. Bundan esa $(0,0)$ muvozanat (maxsus) nuqtaning asimptotik turg'unligi kelib chiqadi. Bu yerda ikki hol bo'lishi mumkin:

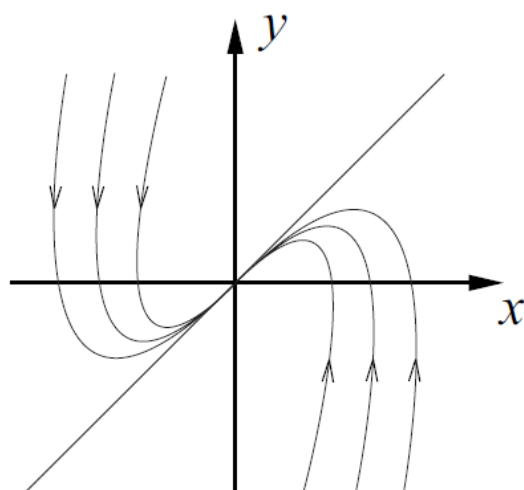
1) $\lambda < 0$ hol:



6-chizma

Bu holda $(0,0)$ muvozanat (maxsus) nuqtaga xos turg'un tugun deyiladi.

2) $\lambda > 0$ hol:



7-chizma

Bu holda $(0,0)$ muvozanat (maxsus) nuqtaga xos noturg'un tugun deyiladi.

e) Aytaylik, A matritsaning xos qiymatlari $\lambda_{1,2} = \mu \pm i\nu$ ko'rinishdagi kompleks son bo'lsin. A matritsaning barcha elementlari haqiqiy sonlardan iborat bo'lgani uchun $\lambda_2 = \bar{\lambda}_1 = \mu - i\nu$. A matritsaning λ_1 xos qiymatiga mos keluvchi xos vektorni $\bar{h} = h_1 - ih_2$ deb belgilaymiz. Bu yerda h_1 va h_2 -haqiqiy vektorlar. U holda $\bar{h} = h_1 + ih_2$ -vektor λ_2 xos qiymatga mos keluvchi xos vektor bo'ladi. (6.3.1) muxtor sistemaning haqiqiy yechimlari

$$x(t) = Ce^{\lambda_1 t} \cdot h + \bar{C}e^{\lambda_2 t} \cdot \bar{h}$$

ko'rinishda bo'ladi. Bu yerda C - ixtiyoriy kompleks son. Agar bu kompleks sonni

$$C = |C|e^{i\varphi}, |C| \geq 0, \varphi = [0, 2\pi); \bar{C} = |C|e^{-i\varphi}$$

ko'rinishda olsak, u holda (6.3.1) sistemaning umumiy yechimi

$$x(t) = 2|C|e^{\mu t} [\cos(\varphi + \nu t)h_1 + \sin(\varphi + \nu t)h_2]$$

ko'rinishni oladi. h_1, h_2 -chiziqli erkli vektorlar bo'lgani uchun ularni R^2 fazoning bazisi sifatida olish mumkin. Agar $x(t)$ yechimning bu bazisdagi koordinatalarini mos ravishda ξ_1 va ξ_2 deb belgilasak,

$$\xi_1(t) = 2|C|e^{\mu t} \cos(\varphi + \nu t), \xi_2(t) = 2|C|e^{\mu t} \sin(\varphi + \nu t)$$

munosabatlar hosil bo'ladi.

Quyidagi

$$r(t) = 2|C|e^{\mu t}, \psi(t) = \varphi + \nu t$$

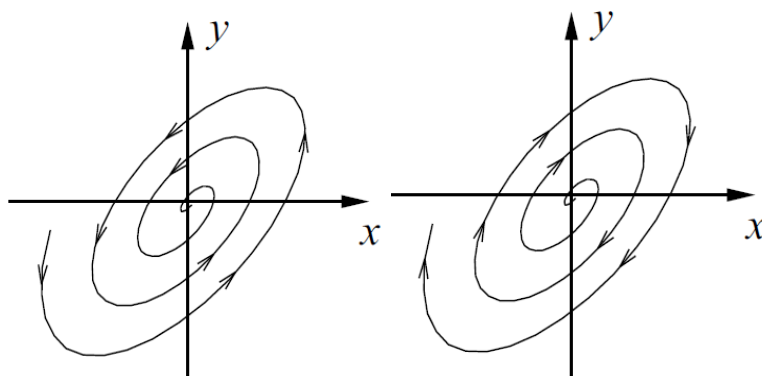
belgilashlarni kiritaylik. Bu belgilashlardan foydalanib (6.3.1) sistemaning trayektoriyalari uchun r, ψ - qutb koordinatalar sistemasida ushbu

$$r = 2|C|e^{\mu \frac{\psi - \varphi}{\nu}}$$

tenglamaga ega bo‘lamiz. Agar $C \neq 0$ va $\mu \neq 0$ bo‘lsa, (6.3.1) sistemaning trayektoriyalari logarifmik spiral shaklidagi egri chiziqlardan iborat bo‘ladi. Agar $\mu = 0$ bo‘lsa, ellips tipidagi egri chiziqlardan iborat bo‘ladi.

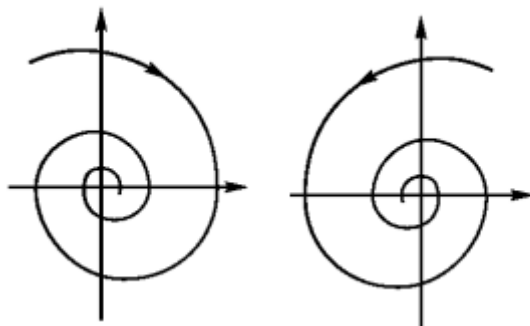
Quyidagi hollarni ko‘rib chiqamiz.

1) Aytaylik, $\mu < 0$ bo‘lsin. U holda $C = 0$ da $x = 0$ muvozanat nuqtaga ega bo‘lamiz. $C \neq 0$ bo‘lsa trayektoriya spiral shaklida bo‘lib harakat muvozanat $(0,0)$ nuqta tomon spiral bo‘yicha harakatlanadi. Chunki, $t \rightarrow +\infty$ da $r(t) \rightarrow +0$, $\psi(t) \rightarrow +\infty$ bo‘ladi. Spiraldagi buralish yo‘nalishi holat $f(x) = Ax$ tezligining yo‘nalishidan aniqlanadi. Masalan, agar x sifatida $(0,1)$ nuqta olinsa, u holda $f(x)$, a_{12} , a_{22} komponentlardan tashkil topgan bo‘ladi. Agar $a_{12} > 0$ bo‘lsa, u holda $f(x)$ o‘nga yo‘nalgan, agar $a_{12} < 0$ bo‘lsa, u holda $f(x)$ chapga yo‘nalgan bo‘ladi. Bu holda $(0,0)$ muvozanat (maxsus) nuqtaga turg‘un fokus deyiladi.



8-chizma

2) Aytaylik, $\mu > 0$ bo‘lsin. U holda $C = 0$ bo‘lganda $x = 0, (0,0)$ muvozanat nuqtaga ega bo‘lamiz. Agar $C \neq 0$ bo‘lsa, u holda $t \rightarrow +\infty$ da nuqta spiral bo‘yicha harakatlanib $x = 0$ dan uzoqlashadi. Chunki $t \rightarrow +\infty$ da $r(t) \rightarrow +\infty$, $\psi(t) \rightarrow +\infty$ bo‘ladi. Bu holda $(0,0)$ muvozanat (maxsus) nuqtaga noturg‘un fokus deyiladi.

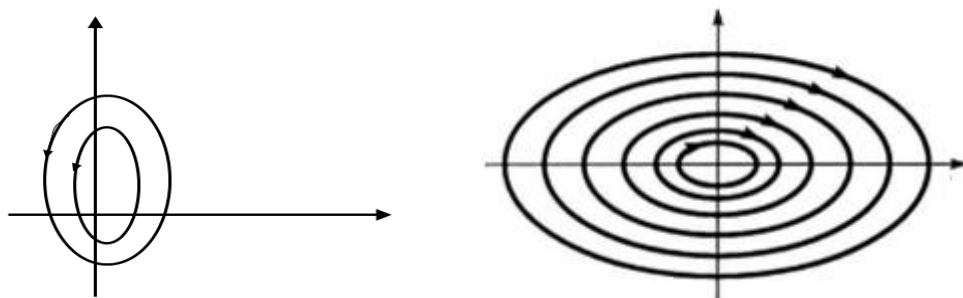


9-chizma

3) Aytaylik, $\mu = 0$, ya‘ni $\lambda_{1,2} = \pm i\nu$ sof mavhum son bo‘lsin. U holda $C \neq 0$ bo‘lganda trayektoriyalar ellips turidagi egri chiziqlardan iborat bo‘ladi. Chunki bu holda (6.3.1) sistemaning umumiy yechimi

$$x(t) = 2|C|[\cos(\varphi + \nu t)h_1 + \sin(\varphi + \nu t)h_2]$$

ko‘rinishda bo‘ladi. Bu yechimga mos keluvchi trayektoriyalar haqiqatdan ham ellipsdan iborat bo‘ladi. $C=0$ da muvozanat nuqtaga ega bo‘lamiz. Qaralayotgan holda $(0,0)$ muvozanat (maxsus) nuqtaga markaz deyiladi. Bu yerda asimptotik turg‘unlik yo‘q, chunki, $(x_1(t), x_2(t)) = x(t)$ nuqta ellipslarning birortasi bo‘ylab cheksiz marta aylanib harakat qiladi. $x_1(t)$ va $x_2(t)$ funksiyalar $t \rightarrow +\infty$ da hech qanday limitga intilmaydi. Chunki, yechim davriy funksiyani ifodalaydi. Ammo $(0,0)$ muvozanat (maxsus) nuqta Lyapunov ma’nosida turg‘un bo‘ladi.



10-chizma

Shunday qilib, oddiy muxtor sistema uchun hammasi bo‘lib 13 ta har xil holatlar fazosi bo‘lishi mumkin ekan.

Mustaqil yechish uchun mashqlar [21], §16, №811-832.

VII BOB. BIRINCHI INTEGRALLAR VA ULARNING TATBIQLARI

1-§. Birinchi integrallar

Aytaylik $G \subset R^n$ sohada ushbu

$$\frac{dx(t)}{dt} = f(x) \quad (7.1.1)$$

muxtor sistema berilgan bo'lsin. Bu yerda $f(x) = (f_1(x), f_2(x), \dots, f_n(x))^T$ - G sohada aniqlangan uzluksiz differensiallanuvchi vektor-funksiya, $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T$ - noma'lum vektor-funksiya, $t \in R$. Faraz qilaylik, $x = \varphi(t)$, $t \in I = (a, b) \subset R$ vektor-funksiya (7.1.1) sistemaning yechimi bo'lsin.

7.1.1-ta'rif. Agar G sohada aniqlangan uzluksiz differensiallanuvchi $U(x)$ funksiya uchun ushbu

$$U[\varphi(t)] \equiv \text{const} \quad (7.1.2)$$

ayniyat (7.1.1) sistemaning har bir yechimi $x = \varphi(t)$, $t \in I$ uchun bajarilsa, $U(x)$ funksiyaga (7.1.1) sistemaning birinchi integrali deyiladi.

Yuqoridagi (7.1.2) ayniyat tarkibidagi o'zgarmas son (7.1.1) sistemaning yechimiga bog'liq va uni topish qiyinchilik tug'dirmaydi. Agar $0 \in (a, b)$ bo'lib, $x = \varphi(t)$ yechim ushbu

$$\varphi(0) = x_0 \in G$$

boshlang'ich shartni qanoatlantirsa, u holda

$$U[\varphi(t)] \equiv U[\varphi(0)] = U(x_0)$$

munosabatning bajarilishi ravshan. Shunday qilib, $U[\varphi(t)]$ ifodaning qiymati faqat (7.1.1) sistema trayektoriyasining tanlanishiga bog'liq bo'lib, t - o'zgaruvchiga bog'liq emas.

Berilgan (7.1.1) sistemaning birinchi integraliga sodda misol sifatida $U(x) = \text{const}$ - o'zgarmas funksiyani olish mumkin.

Ammo (7.1.2), ya'ni $\frac{d}{dt}U[\varphi(t)] = 0$ shartni tekshirish, ancha murakkab masala, chunki (7.1.1) sistemaning yechimini har doim ham topib bo'lavermaydi. Birinchi integralga tekshirishning bir muncha konstruktiv usulini bayon qilamiz.

7.1.2-ta'rif. Uzluksiz differensiallanuvchi $U(x)$, $x \in G$ funksiyaning (7.1.1) muxtor sistema bo'yicha hosilasi deb, ushbu $(f(x), \text{grad}U(x))$ -skalyar ko'paytmaga aytiladi va u $\dot{U}(x)$ orqali belgilanadi:

$$\dot{U}(x) = (f(x), \text{grad}U(x)) = \sum_{j=1}^n f_j(x) \frac{\partial U}{\partial x_j}. \quad (7.1.3)$$

Bu yerda x_j, f_j lar mos ravishda x, f vektorlarning koordinatalari.

Berilgan $U(x)$ funksiyani $\xi \in G$ nuqtada (7.1.1) sistema bo'yicha hosilasini hisoblash uchun, shu sistemaning $\varphi(0, \xi) = \xi$ boshlang'ich shartni qanoatlantiruvchi $\varphi(t, \xi)$ yechimini olib,

$$\left. \frac{d}{dt} U[\varphi(t, \xi)] \right|_{t=0}$$

ifodaning qiymatini topamiz:

$$\begin{aligned} \left. \frac{d}{dt} U[\varphi(t, \xi)] \right|_{t=0} &= \sum_{i=1}^n \frac{\partial U[\varphi(t, \xi)]}{\partial \varphi_i} \cdot \left. \frac{d\varphi_i(t, \xi)}{dt} \right|_{t=0} = \sum_{i=1}^n \frac{\partial U[\varphi(t, \xi)]}{\partial \varphi_i} \cdot f_i(\varphi(t, \xi)) \Big|_{t=0} = \\ &= \sum_{i=1}^n \frac{\partial U(\xi)}{\partial \varphi_i} \cdot f_i(\xi) = \sum_{i=1}^n \frac{\partial U}{\partial x_i} \cdot f_i(x) = \dot{U}(\xi) \end{aligned} \quad (7.1.4)$$

7.1.1-teorema. Uzluksiz differensiallanuvchi $U(x)$, $x \in G$ funksiya (7.1.1) sistemaning birinchi integrali bo'lishi uchun ushbu

$$\dot{U}(x) = 0, \quad \forall x \in G \quad (7.1.5)$$

tenglikning bajarilishi zarur va yetarli.

Isbot. Zarurligi. Aytaylik, $U(x)$ (7.1.1) sistemaning birinchi integrali bo'lsin. U holda ixtiyoriy $\forall \xi \in G$ nuqtada

$$\frac{d}{dt} U(\varphi(t, \xi)) = 0$$

tenglik bajariladi. Bu yerda $x = \varphi(t, \xi)$ funksiya (7.1.1) tenglamaning $x(0) = \varphi(0, \xi) = \xi$ boshlang'ich shartni qanoatlantiruvchi yechimi. Oxirgi tenglikdan va yuqoridagi (7.1.4.) formuladan $\dot{U}(\xi) = 0$ kelib chiqadi.

Yetarliligi. Faraz qilaylik, $\dot{U}(\xi) = 0$, $\forall \xi \in G$ bo'lsin. U holda muxtor sistema yechimining gruppaviy xossasiga ko'ra

$$\varphi(\tau, \varphi(t, \xi)) = \varphi(t + \tau, \xi).$$

Shuning uchun

$$\frac{d}{d\tau} U[\varphi(\tau, \varphi(t, \xi))] = \left. \frac{dU[\varphi(\tau + t, \xi)]}{d\tau} \right|_{\tau=0} = \dot{U}[\varphi(t, \xi)].$$

Teorema shartiga ko'ra $\dot{U}(\xi) = 0$.

Bundan esa

$$\frac{d}{dt} U[\varphi(t, \xi)] = 0$$

kelib chiqadi. Shuning uchun $U(x)$ funksiya (7.1.1) sistemaning birinchi integralidan iborat bo'ladi. Teorema isbot bo'ldi. ■

Ushbu

$$\frac{dx}{dt} = AX$$

ko'rinishdagi chiziqli sistemaning birinchi integrali sifatida quyidagi

$$U(x) = (BX, X)$$

funksiyani olish mumkin. Bunda B -o'z-o'ziga qo'shma, ya'ni $B^* = B$ matritsa bo'lib,

$$(BAX, X) = 0$$

shartni qanoatlantiradi. Bu yerda

$$(X, Y) = \sum_{i=1}^n x_i y_i$$

skalyar ko'paytma. Haqiqatan ham

$$\frac{dU}{dt} = \frac{d}{dt}(BX, X) = (B\dot{X}, X) + (BX, \dot{X}) = 2(BAX, X) = 0.$$

Birinchi integralning geometrik ma'nosi quyidagicha:

Agar $U(x)$ funksiya (7.1.1) sistemaning birinchi integrali bo'lsa, u holda fazoviy trayektoriya $U(x)$ funksiyaning sath sirtlarida yotadi (yoki qism sath sirtlarida yotadi). Agar har bir fazoviy trayektoriya $U(x)$ - birinchi integralning sath sirtlarida yotsa, u holda $U(x)$ ga global birinchi integral deyiladi. Avtonom sistemaning global birinchi integrali har doim ham mavjud bo'lavermaydi. Aniqrog'i shunday muxtor sistemalar borki, ularning o'zgarmsdan farqli birinchi integrallari mavjud emas. Jumladan ushbu

$$\dot{x} = x, \quad \dot{y} = y$$

muxtor sistemani qaraylik. Bu sistemaning o'zgarmsdan farqli birinchi integrali yo'qligini ko'rsatamiz. Aytaylik bu sistemaning birinchi integrali $U(x)$ bo'lsin. Ko'rinib turibdiki $x(t) = C_1 e^t$, $y(t) = C_1 e^t$ berilgan sistemaning trayektoriyalari koordinata boshidan chiquvchi nurlardan iborat bo'ladi. Bu nurlarning har birida $U(x)$ funksiyaning qiymati o'zgarms, ya'ni C ga teng bo'ladi. $x=0$ nuqtada uzluksiz bo'lgani uchun $U(0) = C$. Bundan $U(x)$ funksiyaning o'zgarmsligi kelib chiqadi. Bu esa berilgan sistema faqat trivial $U(x) = C = const$ birinchi integralga ega ekanligini bildiradi. Muxtor sistemaning global bo'lmagan birinchi integraliga lokal birinchi integral deyiladi. Kelgusida biz muxtor sistemaning lokal birinchi integrali hamisha mavjudligini ko'rsatamiz.

7.1.1-misol. Ushbu muxtor sistemaning birinchi integralini toping:

$$\begin{cases} \dot{x}_1(t) = x_2, \\ \dot{x}_2(t) = -x_1 + x_1^3. \end{cases}$$

Yechish. Berilgan sistema tenglamalarini quyidagicha ko'paytiramiz:

$$\dot{x}_1(t) \cdot (-x_1 + x_1^3) = \dot{x}_2(t) \cdot x_2.$$

Bundan

$$\frac{d}{dt} \cdot \left(x_1^2 - \frac{1}{2} x_1^4 + x_2^2 \right) = 0,$$

ya'ni

$$x_1^2 - \frac{1}{2}x_1^4 + x_2^2 = C, \quad C = \text{const}$$

kelib chiqadi. 7.1.1-teoremaga ko'ra,

$$U(x_1, x_2) = x_1^2 - \frac{1}{2}x_1^4 + x_2^2$$

funksiya berilgan sistema uchun birinchi integral bo'ladi. Chunki $\dot{U}(x_1, x_2) = 0$.

Ushbu $x = g(y)$, $y \in G$ silliq teskarilanuvchi almashtirish natijasida G sohada berilgan (7.1.1) muxtor sistema G sohadagi

$$\dot{y}(t) = f_1(y) \equiv [g'(y)]^{-1} \cdot f(g(y)) \quad (7.1.6)$$

muxtor sistemaga o'tadi. Bu yerda

$$g'(y) = \left\| \frac{\partial g_i(y)}{\partial y_j} \right\|, \quad i, j = \overline{1, n}$$

Yakobi matritsasi. Yuqoridagi (7.1.6) tenglamalar sistemasi quyidagi

$$\dot{x} \equiv g'(y) \cdot \dot{y} = f[g(y)]$$

munosabatdan kelib chiqadi. Bu tenglamani \dot{y} ga nisbatan yechish mumkin.

Chunki

$$\det g'(y) = \frac{\partial(g_1, g_2, \dots, g_n)}{\partial(y_1, y_2, \dots, y_n)} \neq 0, \quad y \in G.$$

7.1.1-lemma. Muxtor sistemaning birinchi integrallari silliq almashtirishga nisbatan invariantdir.

Isbot. Aytaylik, $U(x)$ (7.1.1) sistemaning birinchi integrali bo'lsin. Ushbu $x = g(y)$, $y \in G$ silliq teskarilanuvchi almashtirish natijasida G sohada berilgan (7.1.1) muxtor Sistema \tilde{G} sohadagi

$$\dot{y}(t) = \left(\frac{\partial g}{\partial y} \right)^{-1} f(g(y)) \equiv f_1(y) \quad (7.1.7)$$

muxtor sistemaga o'tadi. Bunda

$$\frac{\partial g}{\partial y} = \left\| \frac{\partial g_i(y)}{\partial y_j} \right\|, \quad i, j = \overline{1, n}$$

Yakobi matritsasi. Haqiqatan ham, berilgan (7.1.1) muxtor sistemaga $x = g(y)$ almashtirishni qo'llasak, quyidagi

$$\dot{x} = g'(y) \cdot \dot{y} = f(g(y))$$

munosabat hosil bo'ladi. Bu tenglamani \dot{y} ga nisbatan yechib (7.1.6) tenglikni olamiz. Chunki

$$\det \frac{\partial g}{\partial y} = \det \left\| \frac{\partial g_i}{\partial y_j} \right\| \neq 0; \quad i, j = \overline{1, n}; \quad y \in \tilde{G}.$$

Endi ushbu $V(y)=U(g(y))$ funksiyani (7.1.6) sistemaning birinchi integrali ekanligini ko'rsatamiz. Buning uchun $\dot{V}(y)|_{(7.1.7)}$ funksiyani sistema bo'yicha hosilasini hisoblayiz:

$$\begin{aligned}\dot{V}(y)|_{(7.1.7)} &= \dot{U}(g(y))|_{(7.1.7)} = \sum_{i=1}^n \frac{\partial U}{\partial g_i} \cdot \sum_{j=1}^n \left(\frac{\partial g}{\partial y} \right)_{ij} \cdot \frac{\partial y_j}{\partial t} = \\ &= \sum_{i=1}^n \frac{\partial U}{\partial g_i} \cdot \sum_{j=1}^n \left(\frac{\partial g}{\partial y} \right)_{ij} \cdot \sum_{k=1}^n \left(\frac{\partial g}{\partial y} \right)_{jk}^{-1} f_k(g(y)) = \sum_{i=1}^n \frac{\partial U}{\partial g_i} \cdot f_i(g(y)) = \\ &= \sum_{i=1}^n \frac{\partial U}{\partial x_i} \cdot f_i(x) = (\text{grad}U(x), f(x)) = \dot{U}(x)|_{(6.4.1)} = 0.\end{aligned}$$

Bunda yig'indilarning o'zini almashtirildi va ushbu

$$\sum_{j=1}^n \left(\frac{\partial g}{\partial y} \right)_{ij} \left(\frac{\partial g}{\partial y} \right)_{jk}^{-1} = \delta_i^k = \begin{cases} 0, & i \neq k, \\ 1, & i = k \end{cases}$$

munosabatdan foydalanildi.

Yuqorida olingan tenglikni quyidagicha ham yozish mumkin:

$$\begin{aligned}\dot{V}(y)|_{(7.1.7)} &= (\text{grad}V(y), f_1(y)) = (\text{grad}U(g(y)), [g'(y)]^{-1} f(g(y))) = \\ &= ([g'(y)]^T \cdot \text{grad}U(g(y)), [g'(y)]^{-1} \cdot f(g(y))) = \\ &= (\text{grad}U(g(y)), [g'(y)] \cdot [g'(y)]^{-1} \cdot f(g(y))) = \\ &= (\text{grad}U(x), f(x)) = \dot{U}(x)|_{(7.1.1)} = 0.\end{aligned}$$

Bu yerda $[g'(y)]^T$ orqali transponirlangan Yakobi matritsasi belgilangan. Lemma isbot bo'ldi.

7.1.3-ta'rif. $a \in G$ nuqtaning biror atrofida aniqlangan $U_1(x), U_2(x), \dots, U_k(x), 1 \leq k \leq n$ birinchi integrallar uchun ushbu

$$U'(a) = \left\| \frac{\partial U_i(a)}{\partial x_j} \right\|, \quad i, j = \overline{1, k} - \text{Yakobi matritsasining rangi } k \text{ ga teng bo'lsa, bu}$$

birinchi integrallarga a nuqtada bog'lanmagan deyiladi.

Masalan. (7.1.1) sistemaning $U_1(x) \equiv \text{const}, U_2(x), \dots, U_k(x)$ birinchi integrallari $\forall a \in G$ nuqtada bog'lanmagan bo'lmaydi. Ushbu $U_1(x) = x_1, U_2(x) = x_2, \dots, U_{n-1}(x) = x_{n-1}$ funksiyalar quyidagi

$$\begin{cases} \dot{x}_i = 0, & i = \overline{1, n-1}, \\ \dot{x}_n = 1 \end{cases}$$

sistemaning bog'lanmagan birinchi integrallari bo'ladi.

Quyidagi tasdiq (7.1.1) sistema $(n-1)$ ta bog'lanmagan birinchi integrallari mavjudligining yetarli shartini ifodalaydi (tavsiflaydi).

7.1.3-teorema. Aytaylik, (7.1.1) sistemada $f(a) \neq 0$, $a \in G$ bo'lsin. U holda a nuqtaning biror $G_a \subset G$ atrofida aniqlangan (7.1.1) sistemaning shu nuqtada bog'lanmagan $u_1(x), u_2(x), \dots, u_{n-1}(x)$ birinchi integrallar mavjud. Bundan tashqari, agar $u(x)$ (7.1.1) sistemaning G_a atrofida aniqlangan biror birinchi integrali bo'lsa, u holda shunday uzluksiz differensiallanuvchi $\exists F(\xi_1, \xi_2, \dots, \xi_{n-1})$ funksiya topilib,

$$u(x) = F(u_1(x), u_2(x), \dots, u_{n-1}(x)), \forall x \in G_a \quad (7.1.8)$$

o'rinli bo'ladi.

Isbot. Qaralayotgan $a \in G_a$ nuqta muvozanat nuqta bo'lmagani ($f(a) \neq 0$) uchun, oltinchi bobning birinchi paragrafdagi 6.1.8-teoremaga asosan, uning shunday $\exists G_a$ atrofi va silliq teskarilanuvchi $x = g(y)$ almashtirish topilib, (7.1.1) sistemani quyidagi

$$\begin{cases} \dot{y}_i = 0, i = \overline{1, n-1} \\ \dot{y}_n = 1 \end{cases} \quad (7.1.9)$$

ko'rinishga keltirish mumkin. Bu almashtirish natijasida hosil bo'lgan (7.1.9) sistemaning trayektoriyalari $y_i = C_i, i = \overline{1, n-1}; y_n = t$ tenglamalar orqali aniqlanadi. Ko'rib turibdiki, $v_1(y) = y_1, v_2(y) = y_2, \dots, v_{n-1} = y_{n-1}$ funksiyalar (7.1.9) sistemaning bog'lanmagan birinchi integrallaridan iborat bo'ladi. 7.1.2-teoremaga ko'ra quyidagi

$$u_1(x) = g_1^{-1}(x), u_2(x) = g_2^{-1}(x), \dots, u_{n-1}(x) = g_{n-1}^{-1}(x)$$

funksiyalar berilgan (7.1.1) sistemaning G_a atrofida aniqlangan birinchi integrallari bo'ladi. Bu yerda $y = g^{-1}(x)$ - ushbu $x = g(y)$ funksiyaning teskari funksiyasi bo'lib, uning koordinatalari

$$y_1 = g_1^{-1}(x), y_2 = g_2^{-1}(x), \dots, y_{n-1} = g_{n-1}^{-1}(x).$$

Quyidagi

$$\det[g^{-1}(a)]' = 1: \det g'(a) \neq 0$$

yakobyana noldan farqli bo'lgani sababli, (7.1.1) sistemaning $u_1(x), u_2(x), \dots, u_{n-1}(x)$ birinchi integrallari a nuqtada bog'lanmagan bo'ladi. Yuqoridagi (7.1.9) sistemaning ixtiyoriy birinchi integrali

$$v(y) = F(y_1, y_2, \dots, y_{n-1}) = F[v_1(y), v_2(y), \dots, v_{n-1}(y)]$$

ko'rinishda bo'ladi. Bunda $F, y_i \in \mathbb{R}, i = \overline{1, n-1}$ o'zgaruvchilarning ixtiyoriy uzluksiz differensiallanuvchi funksiyasi. 7.1.2-teoremaga ko'ra $x = g(y)$ almashtirish natijasida (7.1.1) sistema birinchi integralining umumiy ko'rinishiga ega bo'lamiz:

$$u(x) = u[g(y)] = V(y) = V[g^{-1}(x)] = F[g_1^{-1}(x), g_2^{-1}(x), \dots, g_{n-1}^{-1}(x)] =$$

$$= F[u_1(x), u_2(x), \dots, u_{n-1}(x)], x \in G_a. \quad \blacksquare$$

Muxtor sistemaning birinchi integrallari nimaga kerak degan savolning tugʻilishi tabiiy. Bu savolga qisman quyidagi tasdiq javob beradi.

7.1.3-teorema. Faraz qilaylik, (7.1.1) muxtor sistema oʻzgarmasdan farqli birinchi integralga ega boʻlsin. U holda bu sistemaning tartibini bittaga pasaytirish mumkin.

Isbot. Aytaylik, $U(x)$ (7.1.1) sistemaning birinchi integrali boʻlsin. Bundan tashqari $a \in G$ nuqtani shunday tanlaymizki, ushbu $\frac{\partial U}{\partial x_j}(a)$ sonlarning kamida bittasi noldan farqli boʻlsin. Bunday a nuqta mavjud, aks holda $U(x)$ - trivial birinchi integral boʻlar edi. Faraz qilaylik, $\frac{\partial U(a)}{\partial x_n} \neq 0$ boʻlsin. U holda oshkormas funksiyani mavjudligi haqidagi teoremaga asosan $U(x_1, x_2, \dots, x_n) = C$ tenglamani x_n ga nisbatan yechib $x_n = g(x_1, x_2, \dots, x_{n-1}, C)$ munosabatni olish mumkin. Bunda g - birorta silliq funksiya. Bu tenglikning ikki tomonidan (7.1.1) sistema boʻyicha hosila olsak, ushbu

$$f_n(x) - \sum_{j=1}^{n-1} \frac{\partial g}{\partial x_j} f_j(x) = 0 \quad (7.1.10)$$

tenglik hosil boʻladi. Qulaylik uchun $\bar{x} = (x_1, x_2, \dots, x_{n-1})$, $\bar{f} = (f_1, f_2, \dots, f_{n-1})$ deb olamiz. Agar (7.1.1) sistemaning n - tenglamasini ushbu $x_n = g(\bar{x}, C)$ tenglik bilan almashtirsak, u holda uni

$$\dot{\bar{x}} = \bar{f}(\bar{x}, g(\bar{x}, C)) \quad (7.1.11)$$

koʻrinishda yozish mumkin. Koʻrinib turibdiki, bu sistemaning tartibi $n-1$ ga teng.

Agar $\bar{x} = \bar{x}(t)$ funksiya (7.1.10) sistemaning yechimi boʻlsa, u holda $x = x(t)$, $x = (\bar{x}, x_n)$, $x_n(t) = g(\bar{x}(t), C)$ funksiya berilgan (7.1.1) sistemaning yechimi boʻladi. Bu fikrni isbotlash uchun $x_n = g(\bar{x}, C)$ ni quyidagi

$$\frac{dx_n}{dt} = f_n(x) \quad (7.1.11)$$

tenglamani qanoatlantirishini koʻrsatish yetarli. Buning uchun $\frac{dx_n}{dt} = \dot{x}_n(t)$ hosilani hisoblaymiz:

$$\frac{dx_n}{dt} = \sum_{j=1}^{n-1} \frac{\partial g}{\partial x_j} \cdot \frac{dx_j}{dt} = \sum_{j=1}^{n-1} \frac{\partial g(\bar{x}, C)}{\partial x_j} \cdot f_j(\bar{x}, g(\bar{x}, C)) = \sum_{j=1}^{n-1} \frac{\partial g}{\partial x_j} f_j(x). \quad (7.1.12)$$

Yuqoridagi (7.1.10) va (7.1.12) tengliklardan (7.1.11) kelib chiqadi.

Shunday qilib (7.1.1) n ta differensial tenglamalar sistemasining bitta birinchi integrali ma'lum bo'lsa, uning tartibini bittaga kamaytirish mumkin ekan. Teorema isbot bo'ldi. ■

Mustaqil yechish uchun mashqlar [8], §19, №771-775; [21], §19, №941-943; [13], §16, №1-27.

2-§. Gamelton tenglamalar sistemasi va Liuvill teoremasi

Agar differensial tenglamalar sistemasining yechimini ma'lum funksiyalar va ularning boshlang'ich funksiyasi orqali chekli algebraik amallarni qo'llash natijasida topish mumkin bo'lsa, bunday sistemalarga kvadraturada integrallanuvchi deyiladi.

Ushbu $y = (q, p)$, $q = (q_1, q_2, \dots, q_n)$, $p = (p_1, p_2, \dots, p_n)$ ko'rinishdagi $2n$ koordinatali $\{y\}$ vektorlardan tuzilgan R^{2n} fazoni qaraylik.

7.2.1-ta'rif. Ikkita silliq $F, G: E \subset R^{2n} \rightarrow R$ funksiyalarning Puasson qavsi deb quyidagi

$$\Phi = \{F, G\} = \sum_{j=1}^n \left(\frac{\partial F}{\partial q_j} \frac{\partial G}{\partial p_j} - \frac{\partial F}{\partial p_j} \frac{\partial G}{\partial q_j} \right)$$

tenglik yordamida aniqlangan $\Phi: E \rightarrow R$ funksiyaga aytiladi.

Silliq funksiyalardan tuzilgan Puasson qavsi quyidagi:

1) $\{F_1, F_2\} = -\{F_2, F_1\}$,

2) $\{F_1, \{F_2, F_3\}\} + \{F_2, \{F_3, F_1\}\} + \{F_3, \{F_1, F_2\}\} = 0$

munosabatlarni qanoatlantirishini tekshirish qiyinchilik tug'dirmaydi.

Yuqoridagi tengliklarning birinchisi Puasson qavsining antikommutativligini bildiradi. Ikkinchisi esa Yakobi ayniyati deb ataladi.

Ikkita 1) va 2) xossaga ega bo'lgan algebraik amallar kiritilgan vektor fazoga Li algebrasi deyiladi. Shunday qilib Puasson qavsi E da aniqlangan silliq funksiyalar to'plamini Li algebrasiga aylantiradi.

Bundan tashqari Puasson qavsi yana bir shartni qanoatlantiradi:

3) $\{F_1, F_2 F_3\} = \{F_1, F_2\} F_3 + \{F_1, F_3\} F_2$.

Bu esa ko'paytma hosilasi uchun Leybnis formulasini eslatadi.

Ushbu

$$\frac{dy_i}{dt} = \{y_i, H\}, \quad i = \overline{1, 2n} \quad (7.2.1)$$

ko'rinishdagi differensial tenglamalar sistemasiga Gameltonning kanonik tenglamalar sistemasi deyiladi.

Agar $y = (q, p)$ ekanligini e'tiborga olsak, u holda (7.2.1) kanonik sistemani quyidagicha yozish mumkin:

$$\dot{q}_i = \frac{\partial H(q, p)}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H(q, p)}{\partial q_i}. \quad (7.2.2)$$

Bu yerda $H(q, p)$ ga (7.2.1) sistemaning Gamelton funksiyasi yoki Gameltoniyani deyiladi.

Gamelton tenglamalar sistemasiga misol sifatida ushbu

$$m\ddot{x} + F(x) = 0$$

Nyuton tenglamasini olish mumkin. Haqiqatan ham, $q = x$, $p = \dot{x}$ deb olsak quyidagi

$$\begin{cases} \dot{p} = -\frac{F(q)}{m}, \\ \dot{q} = p \end{cases}$$

sistemaga kelamiz. Bunda Gameltoniyani ushbu

$$H = \frac{p^2}{2} + \int_0^q \frac{F(\tau)}{m} d\tau$$

tenglik bilan aniqlanadi.

7.2.2-ta'rif. Agar F va G funksiyalarning Puasson qavsi

$$\{F, G\} = 0$$

tenglikni qanoatlantirsa, bu funksiyalar involyutsiyada deyiladi.

7.2.1-teorema. (Liuvill). (7.2.2) ko'rinishdagi sistemani qaraylik.

Aytaylik $E \subset R^{2n}$ sohada aniqlangan ikki marta uzluksiz differensiallanuvchi bog'lanmagan n ta funksiyalar $F_1, F_2, \dots, F_n : E \rightarrow R$ berilgan bo'lib, ular involyutsiyada

$$\{F_i, F_j\} = 0, \quad F_1 = H$$

bo'lsin. U holda (7.2.2) Gamelton sistemasi kvadraturada integrallanuvchi bo'ladi.

Isbot. Avvalo F_j funksiyalar (7.2.2) sistemaning birinchi integrallaridan iborat bo'lishini ko'rsatamiz. Haqiqatan ham, agar $q = q(t)$, $p = p(t)$ funksiyalar (7.2.2) sistemaning yechimi bo'lsa, u holda

$$\frac{d}{dt} F_j(q(t), p(t)) = \sum_{i=1}^n \left(\frac{\partial F_j}{\partial q_i} \dot{q}_i + \frac{\partial F_j}{\partial p_i} \dot{p}_i \right) = \sum_{i=1}^n \left(\frac{\partial F_j}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial F_j}{\partial p_i} \frac{\partial H}{\partial q_i} \right) = \{F_j, H\} = 0$$

munosabat bajariladi. F_1, F_2, \dots, F_n funksiyalarning bog'lanmaganligidan, ya'ni ushbu dF_1, dF_2, \dots, dF_n differensiallarning chiziqli erkliligidan,

$\left(\frac{\partial F_i}{\partial p_j}, \frac{\partial F_i}{\partial q_k} \right)$ xususiy hosilalardan tuzilgan $n \times 2n$ o'lchamli matritsaning n -

tartibli xosmas minori mavjud. Aytaylik noldan farqli minor oxirgi n - ustunda joylashgan, ya'ni $\det\left(\frac{\partial F_i}{\partial p_j}\right) \neq 0$ bo'lsin.

Yuqoridagi (7.2.2) sistemada (q, p) erkli o'zgaruvchilardan (α, β) erkli o'zgaruvchilarga o'tamiz. Avvalo $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ lar ushbu $\alpha_i = F_i(q, p)$ formula orqali berilgan bo'lsin. Bundan p_j larni topish mumkin, chunki $\det\left(\frac{\partial F_i}{\partial p_j}\right) \neq 0$. Aytaylik, $p_i = f_i(q, \alpha)$ ko'rinishda bo'lsin.

Endi $w = w(q, \alpha)$ funksiyani shunday tanlaymizki, natijada $f_i = \frac{\partial w}{\partial q_i}$ munosabat o'rinli bo'lsin. Analiz kursidan bizga ma'lumki bunday w funksiyani mavjud bo'lishi uchun

$$\frac{\partial}{\partial q_i} f_j = \frac{\partial^2 w}{\partial q_i \partial q_j} = \frac{\partial^2 w}{\partial q_j \partial q_i} = \frac{\partial}{\partial q_j} f_i$$

shartning bajarilishi zarur va yetarli, ya'ni ushbu $\left(\frac{\partial f_i}{\partial q_j}\right)$ xususiy hosilalardan

tuzilgan matritsaning simmetrik bo'lishi zarur va yetarli. Bu shartni tekshiramiz. Shu maqsadda $F(q, f(q, \alpha)) = \alpha$ tenglikni q argument bo'yicha differensiallaymiz. Natijada ushbu $F_q + F_p f_q = 0$ munosabatga ega bo'lamiz.

Bundan $f_q = -F_p^{-1} F_q$ topamiz. Bu yerda f_q, F_p, F_q mos ravishda quyidagi

$$\left(\frac{\partial f_i}{\partial q_j}\right), \left(\frac{\partial F_i}{\partial p_j}\right), \left(\frac{\partial F_i}{\partial q_j}\right)$$

ko'rinishdagi matritsalaridan iborat. Oxirgi tenglikning ikki tomonini transponerlab ushbu

$$f_q^T = -F_q^T (F_p^{-1})^T$$

tenglikni hosil qilamiz. Shunday qilib biz quyidagi

$$f_q = f_q^T,$$

ya'ni unga ekvivalent bo'lgan

$$F_q F_p^T = F_p F_q^T$$

shartni tekshirishimiz lozim. Oxirgi tenglik F_i birinchi integrallarning evolyutivligidan kelib chiqadi. Haqiqatan ham:

$$F_p F_q^T = \begin{vmatrix} F_{1p_1} & F_{1p_2} & \dots & F_{1p_n} \\ \dots & \dots & \dots & \dots \\ F_{np_1} & F_{np_2} & \dots & F_{np_n} \end{vmatrix} \cdot \begin{vmatrix} F_{1q_1} & F_{2q_1} & \dots & F_{nq_1} \\ \dots & \dots & \dots & \dots \\ F_{1q_n} & F_{2q_n} & \dots & F_{nq_n} \end{vmatrix} = \left(\sum_{l=1}^n F_{jp_l} F_{kq_l} \right)_{j,k=1}^n =$$

$$= \begin{vmatrix} \text{invalyutivlikka} \\ \text{asosan} \end{vmatrix} = \left(\sum_{l=1}^n F_{kp_l} F_{jq_l} \right)_{j,k=1}^n = F_q F_p^T$$

Izlanayotgan $w = w(q, \alpha)$ funksiya quyidagi

$$w(q, \alpha) = \int_{q_0}^q (f, dq)$$

egri chiziqli integral orqali aniqlanishi analizdan bizga ma'lum. Bu yerda

$$(f, dq) = \sum_{i=1}^n f_i dq_i - \text{skalyar ko'paytma.}$$

Quyidagi hosilani hisoblaymiz:

$$\frac{d}{dt} \left(\frac{\partial w}{\partial \alpha_k} \right) = \sum_{i=1}^n \dot{q}_i \frac{\partial^2 w}{\partial q_i \partial \alpha_k} = \left| \dot{q}_i = \frac{\partial H}{\partial p_i}, \frac{\partial^2 w}{\partial q_i \partial \alpha_k} = \frac{\partial P_i}{\partial \alpha_k} \right| =$$

$$= \sum_{i=1}^n \frac{\partial H}{\partial p_i} \frac{\partial P_i}{\partial \alpha_k} = \frac{\partial H}{\partial \alpha_k} = \frac{\partial \alpha_1}{\partial \alpha_k} = \begin{cases} 0, & k \neq 1 \\ 1, & k = 1. \end{cases}$$

Endi t o'zgaruvchi bo'yicha integrallab ushbu

$$\frac{\partial w}{\partial \alpha_1} = t + \beta_1, \quad \frac{\partial w}{\partial \alpha_k} = \beta_k, \quad k = \overline{2, n} \quad (7.2.3)$$

munosabatlarni topamiz. Integrallash natijasida hosil bo'lgan $\beta_1, \beta_2, \dots, \beta_n$ o'zgaruvchilarga qo'shimcha sifatida qabul qilamiz. Yuqoridagi (7.2.3) tengliklardan q_i larni α, β va t lar orqali topish mumkin. Chunki,

$$\det \frac{\partial}{\partial t} \left(\frac{\partial w}{\partial \alpha} \right) = \det \frac{\partial f}{\partial \alpha} = \frac{1}{\det F_p} \neq 0.$$

Shunday qilib, (7.2.3) dan $q_i = q_i(\alpha, \beta, t)$ topamiz. Bu yerda $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n), \beta = (\beta_1, \beta_2, \dots, \beta_n)$. Natijada biz izlayotgan almashtirish

$$P_i = f_i(q, \alpha), \quad q_i = q_i(t, \alpha, \beta)$$

ko'rinishda bo'lishini aniqlaymiz. Bu almashtirish natijasida (7.2.2) Gamelton sistema ushbu

$$\dot{\beta}_i = 0, \quad \dot{\alpha}_i = 0 \quad (7.2.4)$$

ko'rinishdagi sistemaga o'tadi va u osongina yechiladi. Teorema isbotlandi.

Teoremani isbotlash jarayonida quyidagi lemmadan foydalanildi.

7.2.1-lemma. Faraz qilaylik, Liuvill teoremasining barcha shartlari bajarilsin. U holda birinchi integrallarni involuntaryligini saqlovchi va

qaralayotgan Gamelton sistemasini ya'na Gamelton sistemasiga o'tkazuvchi va $E' \subset E$ sohada ushbu

$$\det \left(\frac{\partial F_k(q, p)}{\partial p_j} \right) \neq 0$$

shartni qanoatlantiruvchi almashtirish mavjud.

3-§. Harakat – burchak turidagi o'zgaruvchi

Agar $H = H(q, p)$ Gamelton funksiyasi biror p_i yoki q_i koordinataga bog'liq bo'lmasa, u holda bu koordinataga siklik koordinata deyiladi. Siklik koordinata Gamelton sistemasining tartibini pasaytiradi.

Jumladan, agar $\frac{\partial H}{\partial q_1} = 0$ bo'lsa, u holda $\dot{p}_1 = 0$, ya'ni $p_1(t) = \text{const}$

bo'ladi. Ushbu $q' = (q_2, q_3, \dots, q_n)$, $p' = (p_2, p_3, \dots, p_n)$ belgilashlarni kiritaylik. U holda ushbu $\dot{q}' = H_{p'}$, $\dot{p}' = -H_{q'}$ sistemaning tartibi $2n-2$ ga tengligiga ishonch hosil qilamiz. Bu yerda H Gameltonyan p_1 ga tayinlangan parametr sifatida bog'liq. Bu sistemani yechgandan so'ng $q_1 = q_1(t)$ funksiyani $\dot{q}_1 = H_{p_1}(q', p)$ sistemani integrallash natijasida topamiz.

Liuvill teoremasidagi kabi Gamelton sistemasi n ta bog'lanmagan va involyutsiyada bo'lgan birinchi integrallarga ega bo'lgan holini qaraylik. Aytaylik, R^{2n} fazoda M sirt quyidagi

$$F_1(q, p) = \alpha_1, F_2(q, p) = \alpha_2, \dots, F_n(q, p) = \alpha_n \quad (7.3.1)$$

tenglamalar orqali berilgan bo'lib, har bir (q_i, p_i) koordinata tekislikdagi proyeksiyasi yopiq chiziqdan iborat bo'lsin. Bu yerda $\alpha_1, \alpha_2, \dots, \alpha_n$ - ixtiyoriy tayinlangan o'zgarmaslar. Liuvill teoremasini isbotlash jarayonidagi

lemmaga ko'ra, $\det \left(\frac{\partial F_i}{\partial p_i} \right) \neq 0$ deb hisoblashimiz mumkin. Bu shartdan

foydalanib, avvalo (7.3.1) munosabatlardan p ni q va α ning funksiyasi sifatida, ya'ni $p_i = f_i(q, \alpha)$ ko'rinishda aniqlaymiz. So'ngra, agar q_i - siklik o'zgaruvchi bo'lmasa ushbu

$$I_i = \frac{1}{2\pi} \oint_{\gamma_i} p_i dq_i$$

formula orqali yangi o'zgaruvchi kiritamiz. Aks holda, agar q_i - siklik o'zgaruvchi bo'lsa, u holda $I_i = p_i$ tenglik orqali yangi o'zgaruvchini kiritib olamiz. Shunday qilib, yangi I_i o'zgaruvchini quyidagicha aniqlaymiz:

$$I_i = \begin{cases} \frac{1}{2\pi} \oint_{\gamma_i} p_i dq_i, & \text{agar } q_i \text{ – siklik o'zgaruvchi bo'lmasa,} \\ p_i & , \text{ agar } q_i \text{ – siklik o'zgaruvchi bo'lsa.} \end{cases}$$

Bu yerda γ_i M sirtning (q_i, p_i) tekislikdagi proyeksiyasi. Natijada biz

$$I_i = I_i(\alpha) \text{ ko'rinishdagi funksiyani qurib oldik. Faraz qilaylik, } \det\left(\frac{\partial I}{\partial \alpha}\right) \neq 0$$

bo'lsin. U holda oshkormas funksiyaning mavjudligi haqidagi teorema asosan, teskari akslantirish, ya'ni $\alpha_i = \alpha_i(I)$ mavjud. Liuvill teorimasidagi $w = w(q, \alpha)$ funksiyadan foydalanib yangi

$$\omega(q, I) = w(q, \alpha(I))$$

funksiyani aniqlaymiz. Endi dastlabki (q, p) o'zgaruvchilardan yangi

$$\varphi_i = -\frac{\partial \omega}{\partial I_i}, \quad I_i = \frac{1}{2\pi} \oint_{\gamma_i} p_i dq_i, \quad i = \overline{1, n} \quad (7.3.2)$$

o'zgaruvchilarga o'tamiz. Bunda I_i ga “harakat” turidagi o'zgaruvchi, φ_i ga esa “burchak” turidagi o'zgaruvchi deyiladi. Teskari almashtirishni hisoblash uchun, avvalo ushbu

$$p_i = f_i(q, \alpha(I)) = \frac{\partial w(q, \alpha)}{\partial q_i} = \frac{\partial \omega(q, I)}{\partial q_i} \quad (7.3.3)$$

tengliklarni e'tiborga olamiz, so'ngra oshkormas funksiya mavjudligi haqidagi teoremani

$$\varphi_i + \frac{\partial \omega}{\partial I_i} = 0, \quad i = \overline{1, n}$$

munosabatlarga qo'llaymiz. Buning uchun quyidagi Yakobiyanni hisoblaymiz:

$$\begin{aligned} \det\left(\frac{\partial}{\partial q_j} \frac{\partial \omega}{\partial I_i}\right) &= \det\left(\frac{\partial}{\partial I_i} \frac{\partial \omega}{\partial q_j}\right) = \det\left(\frac{\partial}{\partial \alpha_i} \frac{\partial \omega}{\partial q_j}\right) \det\left(\frac{\partial \alpha_i}{\partial I_j}\right) = \\ &= \det\left(\frac{\partial}{\partial \alpha_i} f_j\right) \det\left(\frac{\partial \alpha_i}{\partial I_j}\right) = \left(\det \frac{\partial F_i}{\partial p_j}\right)^{-1} \det\left(\frac{\partial \alpha_i}{\partial I_j}\right) \neq 0. \end{aligned}$$

Yakobiyani noldan farqli bo'lgani uchun teskari funksiya mavjud, ya'ni $q = q(\varphi, I)$.

Endi Gameltoniyanni yangi o'zgaruvchilardagi ko'rinishini topamiz:

$$H(q, p) = H(q(\varphi, I), p(\varphi, I)) = \bar{H}(\varphi, I).$$

Yangi o'zgaruvchini bunday tanlash natijada ushbu

$$\alpha_1(I) = F_1(q, p) = H(q, p) = \bar{H}(\varphi, I),$$

ya'ni $\bar{H}(\varphi, I)$ Gameltonyan "burchak" turidagi o'zgaruvchiga bog'liq bo'lmaydi. Shuning uchun

$$\bar{H} = \bar{H}(I).$$

Endi berilgan tenglamalar sistemasining yangi o'zgaruvchilardagi ko'rinishini aniqlaymiz. Bundan oldin, yangi φ, I o'zgaruvchilarga nisbatan berilgan tenglamalar sistemasini yana $\bar{H} = \bar{H}(\varphi, I)$ gameltonyanli (gamelton tipidagi) tenglamalar sistemasiga o'tishini ko'rsatamiz. (7.3.2) va (7.3.3) tengliklardan

$$p_i = \frac{\partial \omega(q, I)}{\partial q_i}, \quad \varphi_i = -\frac{\partial \omega}{\partial I_i} \quad (7.3.4)$$

topamiz.

Istalgan silliq funksiya uchun

$$d \frac{d\omega}{dt} - \frac{d}{dt} d\omega = 0 \quad (7.3.5')$$

tenglik o'rinli, bu yerda d - funksiyaning $y = (q, p)$ o'zgaruvchi bo'yicha differensial.

Yuqoridagi (7.3.5') formuladan quyidagi

$$-\sum_i \dot{p}_i dq_i + \sum_i \dot{q}_i dp_i - \sum_i \dot{\varphi}_i dI_i + \sum_i \dot{I}_i d\varphi_i = 0 \quad (7.3.5)$$

munosabatga ega bo'lamiz. Ushbu

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

sistemadan

$$dH = -\sum_i \dot{p}_i dq_i + \sum_i \dot{q}_i dp_i \quad (7.3.6)$$

tenglikni hosil qilamiz. $\bar{H} = \bar{H}(\varphi, I)$ funksiya uchun

$$d\bar{H} = \sum_i \frac{\partial \bar{H}}{\partial I_i} dI_i + \sum_i \frac{\partial \bar{H}}{\partial \varphi_i} d\varphi_i \quad (7.3.7)$$

munosabat o'rinli. Yuqoridagi (7.3.5) va (7.3.6) tengliklardan

$$dH = d\bar{H} = \sum_i \dot{\varphi}_i dI_i - \sum_i \dot{I}_i d\varphi_i \quad (7.3.8)$$

kelib chiqadi. (7.3.7) va (7.3.8) tengliklarda $d\varphi_i$ va dI_i differensiallarning bog'lanmaganligidan

$$\dot{\varphi}_i = \frac{\partial \bar{H}}{\partial I_i}, \quad \dot{I}_i = -\frac{\partial \bar{H}}{\partial \varphi_i} \quad (7.3.9)$$

kelib chiqadi. Bu esa sistemaning φ, I o'zgaruvchilarga nisbatan Gamelton turida ekanligini bildiradi. $\bar{H}(\varphi, I)$ Gamelton funksiyasi φ -

o‘zgaruvchilarga bog‘liq emasligini, ya’ni $\bar{H} = \bar{H}(I)$ ekanligini inobatga olsak, (7.3.9) sistema

$$\dot{I}_i = 0, \dot{\varphi}_i = v_i(I)$$

ko‘rinishni oladi. Bundan

$$I_i = const, \varphi_i = v_i(I)t + const$$

kelib chiqadi. Shunday qilib harakat $2n$ - o‘lchamli “harakat-burchak” o‘zgaruvchili fazodagi tor sirtida sodir bo‘lar ekan.

4-§. Kanonik almashtirishlar

Oldingi paragrafda Gamelton turidagi sistemani sodda ko‘rinishga keltiruvchi almashtirish (o‘zgaruvchilarni almashtirish) qurilgan edi. Bunda asosiy vazifani $\omega = \omega(q, I)$ funksiya o‘tagan edi.

7.4.1-ta’rif. Agar quyidagi

$$P_k = \frac{\partial S}{\partial q_k}, \beta_k = \frac{\partial S}{\partial \alpha_k}$$

shartlarni qanoatlantiruvchi silliq $S = S(q, \alpha)$ funksiya topilsa, u holda $P_k = P_k(\alpha, \beta), q_k = q_k(\alpha, \beta)$ koordinatalarga kanonik almashtirish deyiladi. Bunda $S = S(q, \alpha)$ ga hosil qiluvchi funksiya deyiladi.

7.4.1-lemma. Kanonik almashtirishga nisbatan Gamelton tenglamalar sistemasi invariant.

Isbot. Lemmani isbotlash uchun oldingi paragrafdagi (7.3.4), (7.3.9) formulalarga kelish jarayonini takrorlash yetarli. Bunda $S \rightarrow \omega, \beta \rightarrow \varphi, \alpha \rightarrow I$ almashtirishlarni bajarish lozim.

Ravshanki, oldingi paragrafdagi $(p, q) \rightarrow (I, \varphi)$ almashtirish – kanonik almashtirishdan iborat bo‘ladi. Bunda hosil qiluvchi funksiya vazifasini $\omega(q, I)$ o‘taydi.

Ushbu

$$\dot{q}_i = \frac{\partial H(q, p)}{\partial p_i}, \dot{p}_i = -\frac{\partial H(q, p)}{\partial q_i}$$

sistemani sodda ko‘rinishga keltiruvchi kanonik almashtirishni qurishda hosil bo‘lgan sistemaning gameltonyani bitta o‘zgaruvchiga (masalan α) bog‘liq bo‘lib qolgan o‘zgaruvchilarga (β ga) bog‘liq bo‘lmasligi lozim. Agar biz shunday gameltonyanni topsak, u holda quyidagi

$$\begin{cases} \dot{\alpha}_k = -\frac{\partial \bar{H}}{\partial \beta_k} = 0, \\ \dot{\beta}_k = -\frac{\partial \bar{H}}{\partial \alpha_k} = const = \gamma_k \end{cases} \quad (7.4.1)$$

sistemaga ega bo'lamiz. Bundan

$$\alpha_k = \text{const}, \beta_k = \gamma_k t + S_k \quad (7.4.2)$$

kelib chiqadi.

Shunday qilib, $S(q, \alpha)$ hosil qiluvchi funksiyani shunday tanlaymizki, ushbu

$$P_k = \frac{\partial S}{\partial q_k}, \beta_k = \frac{\partial S}{\partial \alpha_k}$$

almashtirish $H(q, p)$ Gameltonyanni $\bar{H}(\alpha)$ Gameltonyanga o'tkazsin. Boshqacha aytganda, $S(q, \alpha)$ funksiya quyidagi

$$H\left(q, \frac{\partial S}{\partial q}\right) = \bar{H}(\alpha)$$

xususiy hosilali tenglamani qanoatlantirsin. Bu tenglamaga Gamelton-Yakobi tenglamasi deyiladi. Kanonik almashtirishni qurish uchun Gamelton-Yakobi tenglamasining birorta xususiy yechimini topish zarur.

7.4.1-misol. Garmonik ossilyator tenglamasi.

Ushbu

$$H(q, p) = \frac{1}{2} p^2 + \frac{aq^2}{m} \quad (7.4.3)$$

ko'rinishdagi Gamelton funksiyaga ega bo'lgan

$$\dot{q} = \frac{\partial H}{\partial p}, \dot{p} = -\frac{\partial H}{\partial q} \quad (7.4.4)$$

tenglamalar sistemasini qaraylik. Bu holda kanonik almashtirishni qurish uchun quyidagi

$$H\left(q, \frac{\partial S}{\partial q}\right) = \bar{H}(\alpha)$$

Gamelton-Yakobi tenglamalaridan $S(q, \alpha)$ yo'naltiruvchi funksiyani topamiz.

Aytaylik, $\bar{H}(\alpha) = \alpha$ bo'lsin, u holda

$$\left(\frac{\partial S}{\partial q}\right)^2 + \frac{aq^2}{m} = \alpha \quad (7.4.5)$$

tenglama hosil bo'ladi. Bundan

$$S = \int_{q_0}^q \sqrt{\alpha - \frac{aq^2}{m}} dq \quad (7.4.6)$$

formulani topamiz. Biz izlayotgan kanonik almashtirish quyidagi

$$\beta = \frac{\partial S}{\partial \alpha} = \frac{1}{2} \int_{q_0}^q \frac{dq}{\sqrt{\alpha - \frac{aq^2}{m}}} \quad (7.4.7)$$

$$P = \frac{\partial S}{\partial q} = \sqrt{\alpha - \frac{aq^2}{m}} \quad (7.4.8)$$

formulalar orqali topiladi. Tanlanishiga ko'ra, $\bar{H}(\alpha) = \alpha$ bo'lgani uchun, yangi o'zgaruvchilarga nisbatan Gamelton sistemasi

$$\dot{\alpha} = \frac{\partial \bar{H}(\alpha)}{\partial \beta} = \frac{\partial \alpha}{\partial \beta} = 0,$$

$$\dot{\beta} = \frac{\partial \bar{H}}{\partial \alpha} = \frac{\partial \alpha}{\partial \alpha} = 1$$

ko'rinishni oladi. Bu sistemani yechib,

$$\alpha = \text{const}, \quad \beta = t - t_0$$

topamiz. Bu yerda t_0 - biror o'zgaruvchi son. Endi teskari $(\alpha, \beta) = (q, p)$ alamshtirish bajaramiz. Buning uchun (7.4.7) integralni hisoblaymiz:

$$\beta = t - t_0 = \sqrt{\frac{m}{a}} \arcsin q \sqrt{\frac{a}{m\alpha}} - \sqrt{\frac{m}{a}} \arcsin q_0 \sqrt{\frac{a}{m\alpha}}.$$

Bundan va (7.4.8) munosabatdan foydalanib,

$$q = \sqrt{\frac{m\alpha}{a}} \sin \left[\sqrt{\frac{a}{m}} (t - t_0) + \arcsin \sqrt{\frac{a}{m\alpha}} q_0 \right],$$

$$p = \alpha \cos \left[\sqrt{\frac{a}{m}} (t - t_0) + \arcsin \sqrt{\frac{a}{m\alpha}} q_0 \right]. \quad (7.4.9)$$

yechimni hosil qilamiz.

5-§. Xususiy hosilali birinchi tartibli bir jinsli differensial tenglamalar

Ikki o'zgaruvchili $u(x, y)$ funksiya biror $D \subset R^2$ sohada aniqlangan bo'lsin. Bu yerda x va y lar erkli o'zgaruvchilar.

7.5.1-ta'rif. Erkli o'zgaruvchi x va y hamda noma'lum funksiya u va uning xususiy hosilalari $u'_x = \frac{\partial u}{\partial x}$, $u'_y = \frac{\partial u}{\partial y}$ orasidagi ushbu

$$F(x, y, u, u'_x, u'_y) = 0 \quad (7.5.1)$$

funksional bog'lanishga birinchi tartibli xususiy hosilali differensial tenglama deyiladi.

Quyidagi xususiy hosilali tenglamalarni ko'rib chiqamiz:

1. Ushbu

$$A(x, y) \frac{\partial u}{\partial x} + B(x, y) \frac{\partial u}{\partial y} + f(x, y, u) = 0 \quad (7.5.2)$$

ko‘rinishdagi tenglamaga birinchi tartibli xususiy hosilali kvazichiziqli differensial tenglama deb ataladi.

2. Ushbu

$$A(x, y) \frac{\partial u}{\partial x} + B(x, y) \frac{\partial u}{\partial y} + C(x, y)u + D(x, y) = 0 \quad (7.5.3)$$

ko‘rinishdagi tenglamaga birinchi tartibli chiziqli bir jinsli bo‘lmagan xususiy hosilali differensial tenglama deyiladi.

Agar (7.5.3) tenglamada $D(x, y) = 0$ bo‘lsa, u holda ushbu

$$A(x, y) \frac{\partial u}{\partial x} + B(x, y) \frac{\partial u}{\partial y} + C(x, y)u = 0 \quad (7.5.3')$$

ko‘rinishdagi tenglamaga birinchi tartibli bir jinsli chiziqli xususiy hosilali differensial tenglama deyiladi.

Soddalik uchun $C(x, y) = 0$ holni, ya’ni

$$A(x, y) \frac{\partial u}{\partial x} + B(x, y) \frac{\partial u}{\partial y} = 0 \quad (7.5.4)$$

ko‘rinishdagi birinchi tartibli bir jinsli ikki o‘zgaruvchili xususiy hosilali chiziqli differensial tenglamani qaraylik.

Aytaylik, $R^2 \supset D$ sohada $A(x, y), B(x, y) \in C^1(D)$ funksiyalar bir marta uzluksiz differensiallanuvchi bo‘lib, $A^2(x, y) + B^2(x, y) > 0$ shartni qanoatlantirsin.

Yuqoridagi (7.5.4) tenglamaga mos quyidagi

$$\frac{dx}{A} = \frac{dy}{B} = dt \quad (7.5.5)$$

oddiy differensial tenglamalar sistemasini tuzib olaylik. U holda (7.5.5) differensial tenglamalar sistemasiga (7.5.4) xususiy hosilali differensial tenglamaning xarakteristik tenglamasi deyiladi.

Faraz qilaylik,

$$x = x(t), \quad y = y(t) \quad (7.5.6)$$

funksiyalar (7.5.5) oddiy differensial tenglamalar sistemasining integrali (yechimi) bo‘lsin.

Agar biror $\psi(x, y)$ funksiya uchun ushbu

$$\psi(x(t), y(t)) = C = \text{const}$$

ayniyat (7.5.5) sistemaning har bir integral chizig‘i uchun bajarilsa, u holda quyidagi

$$\psi(x(t), y(t)) = C$$

tenglikka (7.5.5) sistemaning birinchi integrali deyiladi.

7.5.1-teorema. Ushbu $\omega(x, y) = \psi(x, y)$ funksiya (7.5.4) xususiy hosilali bir jinsli differensial tenglamaning yechimi bo‘lishi uchun

$$\psi(x, y) = C$$

ifoda (7.5.5) sistemaning birinchi integralidan iborat bo'lishi zarur va yetarli.

Isbot. Zaruriyligi. Aytaylik, $\omega = \psi(x, y)$ funksiya (7.5.4) xususiy hosilali differensial tenglamaning yechimi bo'lsin, ya'ni

$$A \frac{\partial \psi}{\partial x} + B \frac{\partial \psi}{\partial y} \equiv 0$$

ayniyat bajarilsin. Ikkinchi tomondan (7.5.6) integral chiziqlar ustida ψ funksiya t parametrغا bog'liq bo'ladi, ya'ni

$$\psi(x(t), y(t)) = \psi(t)$$

o'rinli bo'ladi. Bu tenglikning ikkala tomonini t o'zgaruvchi bo'yicha differensiallaymiz:

$$\frac{d\psi}{dt} = \frac{\partial \psi}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial \psi}{\partial y} \cdot \frac{dy}{dt} \quad (7.5.7)$$

Bunda $x = x(t)$, $y = y(t)$ funksiyalar (7.5.5) sistemaning yechimi bo'lgani uchun

$$\frac{dx}{dt} = A; \quad \frac{dy}{dt} = B$$

munosabatlar bajariladi. Bundan foydalanib (7.5.7) tenglamani quyidagicha yozish mumkin:

$$\frac{d\psi}{dt} = \frac{\partial \psi}{\partial x} A + \frac{\partial \psi}{\partial y} B \equiv 0,$$

ya'ni

$$\frac{d\psi}{dt} = 0.$$

Bundan esa o'z navbatida

$$\psi(x(t), y(t)) = \text{const}$$

ekanligi kelib chiqadi. Oxirgi tenglik (7.5.5) sistemaning birinchi integralini ifodalaydi.

Yetarliligi. Faraz qilaylik, $\psi(x, y) = C$ (7.5.5) sistemaning birinchi integralidan iborat bo'lsin. U holda (7.5.6) integral chiziqlar ustida $\psi(x(t), y(t)) = C$ tenglik bajariladi. Bu tenglikning ikkala tomonini differensiallab,

$$d\psi = \frac{\partial \psi}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial \psi}{\partial y} \cdot \frac{dy}{dt} = dC = 0 \quad (7.5.8)$$

munosabatni topamiz. Bunda $x = x(t)$, $y = y(t)$ funksiyalar (7.5.5) sistemaning integrali, ya'ni yechim bo'lgani uchun

$$\frac{dx}{dt} = A; \quad \frac{dy}{dt} = B$$

o'rinli. Bundan ko'rinadiki, $\omega = \psi(x, y)$ funksiya (7.5.4) xususiy hosilali differensial tenglamaning yechimi bo'lar ekan, ya'ni

$$\frac{\partial \psi}{\partial x} A + \frac{\partial \psi}{\partial y} B = 0.$$

Shunday qilib, biz (7.5.4) ko‘rinishdagi xususiy hosilali differensial tenglama bilan (7.5.5) ko‘rinishdagi ushbu

$$\frac{dx}{dt} = A; \quad \frac{dy}{dt} = B$$

oddiy differensial tenglamalar orasida uzviy bog‘lanish borligini ko‘rsatdik.

7.5.1-izoh. Agar $\psi(x, y) = C$ (7.5.5) sistemaning birinchi integralidan iborat bo‘lsa, u holda (7.5.4) xususiy hosilali differensial tenglamaning umumiy yechimi

$$u(x, y) = f(\psi(x, y))$$

ko‘rinishga keladi. Bu yerda f - ixtiyoriy uzluksiz differensiallanuvchi funksiya, ya’ni $f \in C^1$.

Misollar yechishga namunalar

1. Ushbu

$$x \frac{\partial u}{\partial x} - 2y \frac{\partial u}{\partial y} = 0$$

bir jinsli xususiy hosilali differensial tenglamaning yechimi topilsin.

Yechish. Avvalo, berilgan xususiy hosilali differensial tenglamaga mos keluvchi oddiy differensial tenglamalar sistemasini tuzib olamiz:

$$\frac{dx}{x} = \frac{dy}{-2y} = dt$$

Endi, quyidagi

$$\frac{dx}{x} = dt, \quad \frac{dy}{-2y} = dt$$

oddiy differensial tenglamalarning yechimlarini topamiz:

$$\int \frac{dx}{x} = \int dt, \quad x(t) = Ce^t,$$

$$\int \frac{dy}{-2y} = -\int dt, \quad \sqrt{y(t)} = C_1 e^{-t}.$$

Topilgan bu yechimlarni o‘zaro ko‘paytirib,

$$x(t)\sqrt{y(t)} = \bar{C}, \quad \bar{C} = C \cdot C_1 = \text{const}$$

tenglikni hosil qilamiz. Demak,

$$\psi(x, y) = x\sqrt{y} = C$$

funksiya

$$\frac{dx}{x} = \frac{dy}{-2y} = dt$$

oddiy differensial tenglamalar sistemasining birinchi integralidan iborat bo‘lar ekan. Shuning uchun

$$u(x, y) = x\sqrt{y}$$

funksiya berilgan xususiy hosilali differensial tenglamaning yechimi bo‘ladi.

Yuqoridagi 7.5.1-izohga asosan berilgan xususiy hosilali differensial tenglamaning umumiy yechimi ushbu

$$u(x, y) = f(x\sqrt{y}), \quad \forall f \in C^1$$

ko‘rinishda bo‘ladi.

Agar $f(x) = x$ ko‘rinishda bo‘lsa, u holda $u(x, y) = x\sqrt{y}$ funksiya berilgan differensial tenglamaning yechimi bo‘ladi.

Agar $f(x) = x^2$ ko‘rinishda bo‘lsa, u holda ushbu

$$u(x, y) = (x\sqrt{y})^2 = x^2 y$$

funksiya berilgan xususiy hosilali differensial tenglamaning yechimidan iborat bo‘ladi.

2. Ushbu

$$\frac{\partial u}{\partial x} - y \frac{\partial u}{\partial y} = 0$$

xususiy hosilali bir jinsli differensial tenglamaning yechimi topilsin.

Yechish. Berilgan xususiy hosilali differensial tenglamaga mos keluvchi

$$\frac{dx}{1} = \frac{dy}{-y} = dt$$

oddiy differensial tenglamalar sistemasini tuzib olamiz. Ushbu

$$\int \frac{dx}{1} = \int \frac{dy}{-y}$$

integrallarni hisoblab, uning yechimini topish mumkin:

$$x = -\ln y + \ln C \Rightarrow y \cdot e^x = C, \quad C = \text{const.}$$

Bu esa oddiy differensial tenglamalar sistemasining birinchi integralidir. Demak, quyidagi

$$u(x, y) = ye^x$$

funksiya berilgan xususiy hosilali differensial tenglamaning yechimidan iborat bo‘lar ekan. 7.5.1-izohga asosan berilgan xususiy hosilali differensial tenglamaning umumiy yechimini ham topish mumkin:

$$u(x, y) = f(y \cdot e^x), \quad \forall f \in C^1.$$

6-§. Birinchi tartibli xususiy hosilali chiziqli differensial tenglama uchun Koshi masalasi

I. Birinchi tartibli ikki o'zgaruvchili xususiy hosilali chiziqli differensial tenglamaning umumiy ko'rinishi quyidagicha bo'ladi:

$$a(x, y) \frac{\partial u}{\partial x} + b(x, y) \frac{\partial u}{\partial y} + c(x, y)u = f(x, y). \quad (7.6.1)$$

Bu yerda $a = a(x, y)$, $b = b(x, y)$, $c = c(x, y)$, $f(x, y)$, $D(\subset R^2)$ sohada aniqlangan va uzluksiz differentsiallanuvchi funksiyalar bo'lib, $a(x, y), b(x, y) \neq (0, 0)$, $\forall (x, y) \in D$ shartni qanoatlantiradi. R^2 - tekislikdagi γ - egri chiziq ushbu

$$x = \varphi(s), y = \psi(s), s \in I = (s_1, s_2)$$

ko'rinishdagi tenglamasi bilan berilgan bo'lib,

$$(\varphi'(s), \psi'(s)) \neq (0, 0), \forall s \in I$$

shartni qanoatlantirsin. Boshqacha aytganda, γ - silliq chiziq bo'lsin.

Aytaylik, γ chiziqda $u(x, y)$ - noma'lum funksiyaning qiymati

$$u|_{\gamma} = h(s), \quad (7.6.2)$$

ya'ni

$$u|_{\gamma} = u(\varphi(s), \psi(s)) = h(s), \forall s \in I \quad (7.6.3)$$

berilgan bo'lsin. Bu yerda $h \in C^1(I)$ - berilgan differentsiallanuvchi funksiya.

7.6.1-ta'rif. (7.6.1) xususiy hosilali differensial tenglamaning (7.6.3) boshlang'ich shartni qanoatlantiruvchi $u(x, y)$ yechimini topishga Koshi masalasi deyiladi.

(7.6.1) ko'rinishdagi xususiy hosilali differensial tenglamaga mos keluvchi xarakteristik tenglama ushbu

$$\frac{dx}{dt} = a(x, y), \frac{dy}{dt} = b(x, y) \quad (7.6.4)$$

ko'rinishda bo'ladi. Bu oddiy differensial tenglamalar sistemasining yechimlari (fazoviy trayektoriyalari) (7.6.1) xususiy hosilali differensial tenglamaning xarakteristikalari deyiladi. Xarakteristikalar bilan (7.6.1) xususiy hosilali differensial tenglama o'rtasida uzviy bog'lanish mavjud.

7.6.1-Lemma. Xarakteristika ustida ushbu

$$\frac{du}{dt} = a \frac{\partial u}{\partial x} + b \frac{\partial u}{\partial y} \quad (7.6.5)$$

munosabat o'rinli.

Isbot. Faraz qilaylik,

$$x = x(t), y = y(t)$$

(7.6.4) sistemaning yechimi, ya'ni xarakteristikalar berilgan bo'lsin. U holda quyidagi

$$\frac{d}{dt}u(x(t), y(t)) = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt} = \frac{\partial u}{\partial x} a(x, y) + \frac{\partial u}{\partial y} b(x, y)$$

tenglikka ega bo'lamiz.

7.6.1-teorema. Agar γ - berilgan egri chiziq (7.6.1) xusuiy hosilali differensial tenglamaning xarakteristikalariga urinmasa, u holda (7.6.1)-(7.6.3) Koshi masalasi γ egri chiziqning biror atrofida yagona yechimga ega bo'ladi.

Isbot. Mavjudligi. Berilgan γ egri chiziqning har bir nuqtasidan xarakteristika chiqaramiz, ya'ni (7.6.4) sistema uchun quyidagi Koshi masalasini yechamiz:

$$\begin{cases} \frac{dx}{dt} = a(x, y), \\ \frac{dy}{dt} = b(x, y) \end{cases} \quad (7.6.6)$$

$$x|_{t=0} = \varphi(s), \quad y|_{t=0} = \psi(s), \quad s \in I = (s_1, s_2).$$

Bu masalaning yechimi mavjud va yagona bo'lgani uchun, uni

$$x = x(t, s), \quad y = y(t, s) \quad (7.6.7)$$

ko'rinishda yozish mumkin. Bu xarakteristikani mos ravishda l_s orqali belgilaymiz. 7.6.1-lemmaga ko'ra l_s - xarakteristika ustida (7.6.1) tenglama quyidagi ko'rinishni oladi:

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt} = a(x, y) \frac{\partial u}{\partial x} + b(x, y) \frac{\partial u}{\partial y} = f(x, y) - c(x, y)u,$$

ya'ni

$$\frac{du}{dt} + cu = f \quad (7.6.8)$$

$$u|_{t=0} = h(s)$$

hosil bo'ladi. Shunday qilib, biz (7.6.8) chiziqli oddiy differensial tenglamaga qo'yilgan Koshi masalasiga ega bo'ldik. Endi (7.6.8) masalaning yechimini

$$u = \omega(t, s)$$

orqali belgilymiz va $\omega(t, s)$ funksiyaning x, y o'zgaruvchilarga nisbatan silliqqligini ko'rsatamiz. Buning uchun (7.6.7) tenglamalar sistemasidan t, s larni x va y lar orqali aniqlaymiz. Shu maqsadda quyidagi Yakobianni tekshiramiz:

$$J = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{vmatrix} = \begin{vmatrix} \varphi'(s) & \psi'(s) \\ a(x, y) & b(x, y) \end{vmatrix}_{(x, y) \in \gamma} = b(x, y)\varphi'(s) - a(x, y)\psi'(s) \neq 0.$$

Chunki, $\bar{m} = (\varphi', \psi')$ - vektor γ egri chiziqning urinmasi, ushbu $\bar{p} = (a(x, y), b(x, y))$ -vektor esa l_s xarakteristikaning urinmasidan, iborat. Berilgan γ egri chiziq xarakteristikalarga urinmaganligi sababli bu \bar{m} va \bar{p} vektorlar nokollenyardir.

Endi, oshkormas funksiya mavjudligi haqidagi teorema asosan (7.6.7) dan $t = \xi(x, y)$, $s = \eta(x, y)$ larni topamiz. So'ngra bu topilganlarni (7.6.8) masalaning yechimi $u = \omega(t, s)$ ga qo'yamiz. Natijada ushbu

$$\omega(t, s) = \omega(\xi(x, y), \eta(x, y)) = u(x, y)$$

munosabat olamiz. Bu esa biz izlagan yechimdir.

Yagonaligi. Faraz qilaylik, berilgan masala ikkita $u_1(x, y)$ va $u_2(x, y)$ yechimlarga ega bo'lsin. Quyidagi

$$\tilde{u} = u_1 - u_2$$

belgilashni kiritaylik. Ko'rinib turibdiki, $\tilde{u}(x, y)$ funksiya ushbu

$$a(x, y) \frac{\partial \tilde{u}}{\partial x} + b(x, y) \frac{\partial \tilde{u}}{\partial y} + c(x, y)\tilde{u} = 0, \quad (7.6.9)$$

$$\tilde{u}|_{\gamma} = 0$$

Koshi masalasining yechimidan iborat. 7.6.1-lemmaga ko'ra, (7.6.9) ni quyidagicha yozish mumkin:

$$\frac{d\tilde{u}}{dt} - c\tilde{u} = 0, \quad (7.6.10)$$

$$\tilde{u}|_{t=0} = 0.$$

Bu Koshi masalasi faqat $\tilde{u} = 0$ nol yechimga ega bo'lishi ravshan. Bundan $u_1 = u_2$ kelib chiqadi.

1. γ egri chiziqdan o'tuvchi xarakteristikalarni aniqlash, ya'ni (7.6.6) Koshi masalalari oilasini yechish.

2. Chizikli tenglamaga qo'yilgan (7.6.8) Koshi masalalari oilasini yechish.

Izoh. Agar γ egri chiziq xarakteristikalarning birortasiga urinsa, u holda (7.6.1), (7.6.3) Koshi masalasi yechimga ega bo'lmaydi yoki cheksiz ko'p yechimga ega bo'ladi.

Misollar yechish namunalari

1. Ushbu Koshi masalasining yechimi topilsin:

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0, u(x, y)|_{y=1} = x$$

Yechish. Avvalo berilgan xususiy hosilali tenglamaga mos keluvchi xarakteristik tenglama tuzib olamiz:

$$\begin{cases} \frac{dx}{dt} = x, & x = C_1 e^t \\ \frac{dy}{dt} = y, & y = C_2 e^t \end{cases} \quad \begin{cases} x = C_1 e^t, \\ y = C_2 e^t, \end{cases}$$

bu yerda C_1 va C_2 - ixtiyoriy haqiqiy sonlar.

Bundan

$$\frac{x}{y} = const$$

ekanligi kelib chiqadi. Bu tenglik oddiy differensial tenglamalar sistemasining birinchi integralidan iborat bo'ladi. Shuning uchun berilgan xususiy hosilali differensial tenglamaning umumiy yechimi

$$u(x, y) = f\left(\frac{x}{y}\right), \forall f \in C^1(D)$$

ko'rinishda bo'ladi.

Endi ushbu

$$u(x, 1) = x$$

boshlang'ich shartdan $f(x) = x$ kelib chiqadi. Demak, berilgan Koshi masalasining yechimi

$$u(x, y) = \frac{x}{y}$$

ko'rinishda bo'lar ekan.

2. Ushbu Koshi masalasining yechimini toping:

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0, u(x, y)|_{y=x} = 1.$$

Yechish. Berilgan xususiy hosilali differensial tenglamaning xarakteristik tenglamasi

$$\frac{dx}{dt} = 1, \frac{dy}{dt} = 1,$$

ya'ni

$$\frac{dx}{1} = \frac{dy}{1} = dt$$

ko‘rinishda bo‘ladi. Quyidagi

$$\frac{dx}{1} = \frac{dy}{1}$$

tenglamani integrallab,

$$y - x = C, \quad C = \text{const}$$

xarakteristikalarini topamiz. Bu esa oddiy differensial tenglamalar sistemasining birinchi integralidan iborat bo‘ladi. Shuning uchun berilgan xususiy hosilali differensial tenglamaning umumiy yechimi

$$u(x, y) = f(y - x), \quad \forall f \in C^1$$

ko‘rinishda bo‘ladi. Boshlang‘ich shartlardan foydalanib

$$u(x, x) = f(0) = 1$$

ekanligini topamiz. Demak $f(x)$ - ixtiyoriy differensiallanuvchi funksiya bo‘lib

$$f(0) = 1$$

shartni qanoatlantirar ekan. Agar $f(x) = \cos x$ bo‘lsa, u holda $u(x, y) = \cos(y - x)$ ko‘rinishdagi funksiya berilgan Koshi masalasining yechimidan iborat bo‘ladi. Qaralayotgan holda berilgan Koshi masalasining yechimi cheksiz ko‘p. Chunki, $y - x = 0$ chiziq xarakteristika bo‘ladi.

7.6.1-teoremada ta’kidlanganidek, Koshi masalasining yechimi mavjud va yagona bo‘lishi uchun γ - egri chiziq xarakteristikadan iborat bo‘lmasligi lozim, xattoki xarakteristikaga urinishi ham mumkin emas.

3. Ushbu Koshi masalasining yechimini toping:

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 1, \quad u(x, y)|_{y=1} = x$$

Yechish. Berilgan xususiy hosilali tenglama bir jinsli bo‘lmagani uchun, uning umumiy yechimi

$$u(x, y) = u_0(x, y) + u_{xus}(x, y)$$

ko‘rinishda bo‘ladi. Bu yerda $u_0(x, y)$ funksiya quyidagi

$$x \frac{\partial u_0}{\partial x} + y \frac{\partial u_0}{\partial y} = 0$$

bir jinsli differensial tenglamaning umumiy yechimi. Biz bu differensial tenglamaning umumiy yechimini 1-misolda topgan edik:

$$u_0(x, y) = f\left(\frac{x}{y}\right), \quad \forall f \in C^1$$

Endi $u_{xus}(x, y)$ - xususiy yechimini topish bilan shug‘ullanamiz. Buning uchun

$$\frac{dx}{dt} = x, \quad \frac{dy}{dt} = y$$

xarakteristik tenglamalardan foydalanib, quyidagi hosilalarni hisoblaymiz:

$$\frac{du_{xus}(x(t), y(t))}{dt} = \frac{\partial u_{xus}}{\partial x} \frac{dx}{dt} + \frac{\partial u_{xus}}{\partial y} \frac{dy}{dt} = \frac{\partial u_{xus}}{\partial x} x + \frac{\partial u_{xus}}{\partial y} y = 1$$

Demak, $u_{xus}(x(t), y(t))$ yechim xarakteristikalar ustida ushbu

$$\frac{du_{xus}}{dt} = 1$$

tenglamani qanoatlantirar ekan. Bu tenglamani integrallab,

$$u_{xus} = t$$

ekanligini topamiz. Endi xarakteristik tenglamalarni integrallab, ushbu

$$x(t) = Ce^t \text{ va } y(t) = C_1 e^t$$

xarakteristikalarni topamiz. Bunda boshlang'ich shartdan foydalanib,

$$1 = y(0) = C_1 e^0 = C_1,$$

$$y = e^t, \quad t = \ln y$$

munosabatlarga ega bo'lamiz. Nihoyat,

$$u_{xus} = \ln y$$

hosil bo'ladi. Shunday qilib, berilgan xususiy hosilali differensial tenglamaning umumiy yechimi

$$u(x, y) = f\left(\frac{x}{y}\right) + \ln y$$

ko'rinishda bo'lar ekan. Boshlang'ich shartga ko'ra,

$$u(x, 1) = x, \quad f(x) = x$$

bo'ladi. Bundan foydalanib, biz izlagan Koshi masalasining yechimini topamiz:

$$u(x, y) = \frac{x}{y} + \ln y$$

4. Endi ushbu

$$b_1(x, y) \frac{\partial u}{\partial x} + b_2(x, y) \frac{\partial u}{\partial y} = 0, \quad (7.6.11)$$

$$u(x, y)|_{y=y_0} = \varphi(x) \quad (7.6.12)$$

Koshi masalasini qaraylik. Bu yerda $\varphi(x)$ - berilgan uzluksiz differensiallanuvchi funksiya. Bu Koshi masalasini yechish uchun avvalo, uning xarakteristik tenglamasini tuzib olamiz:

$$\frac{dx}{b_1(x, y)} = \frac{dy}{b_2(x, y)} = dt.$$

So'ngra uning birinchi integralini topamiz:

$$\psi(x, y) = C.$$

Bundan so'ng (7.6.11) tenglamaning umumiy yechimini aniqlaymiz:

$$u(x, y) = f(\psi(x, y)), \forall f \in C^1(D).$$

Endi (7.6.12) boshlang'ich shartdan foydalanib,

$$u(x, y)|_{y=y_0} = f(\psi(x, y_0)) = \varphi(x) \quad (7.6.13)$$

topamiz. Oxirgi

$$f(\psi(x, y_0)) = \varphi(x)$$

tenglikdan f funksiyani topamiz. Bundan esa

$$\psi(x, y_0) = f_{y_0}^{-1}(\varphi(x))$$

topib, berilgan Koshi masalasining

$$u(x, y) = f(f_y^{-1}(\varphi(x))) \quad (7.6.14)$$

yechimini hosil qilamiz.

5. Ushbu

$$x \frac{\partial u}{\partial x} - y \frac{\partial u}{\partial y} = 0$$

tenglamaning

$$u(x, y)|_{y=0} = \varphi(x) \in C^1$$

boshlang'ich shartni qanoatlantiruvchi yechimini toping.

Yechish. Avvalo, berilgan xususiy hosilali differensial tenglamaning xarakteristik tenglamasini tuzamiz:

$$\frac{dx}{y} = \frac{dy}{-x} = dt.$$

So'ngra ushbu

$$x dx = -y dy$$

oddiy differensial tenglamani integrallab,

$$\frac{x^2}{2} = -\frac{y^2}{2} + C, \quad x^2 + y^2 = 2C = C_1$$

uning birinchi integralini topamiz. Bundan foydalanib, berilgan xususiy hosilali differensial tenglamaning umumiy yechimini aniqlaymiz:

$$u(x, y) = f(x^2 + y^2), \forall f \in C^1.$$

Berilgan boshlang'ich shartdan foydalanib,

$$u(x, y)|_{y=0} = \varphi(x) = f(x^2)$$

munosabatni hosil qilamiz. Demak,

$$f(x) = \varphi(\sqrt{x})$$

ko'rinishda bo'lar ekan. Bundan esa

$$u(x, y) = \varphi(\sqrt{x^2 + y^2})$$

Koshi masalasining yechimi kelib chiqadi.

6. Endi quyidagi masalani qaraymiz: ushbu

$$b_1(x, y) \frac{\partial u}{\partial x} + b_2(x, y) \frac{\partial u}{\partial y} = 0 \quad (7.6.15)$$

differentensial tenglamaning quyidagi

$$x = \varphi_1(t), y = \varphi_2(t), u = \varphi(t) \quad (7.6.16)$$

chiziqni o'z ichiga olgan $u(x, y)$ yechimi topilsin. Bunda $\varphi(t) = u(\varphi_1(t), \varphi_2(t))$ ayniyatning bajarilishi lozim. Boshqacha aytganda (7.6.15) tenglamaning berilgan (7.6.16) chiziqdan o'tuvchi integral sirtini topish talab qilinadi.

Yechish. Qaralayotgan masalani yechish uchun xuddi avvalgidek

$$\frac{dx}{b_1(x, y)} = \frac{dy}{b_2(x, y)} = dt \quad (7.6.17)$$

xarakteristik tenglama tuzib olamiz.

Aytaylik,

$$\psi(x, y) = C \quad (7.6.18)$$

(7.6.17) oddiy differentensial tenglamaning birinchi integrali bo'lsin.

Faraz qilaylik, $x = \varphi_1(t)$, $y = \varphi_2(t)$ egri chiziq (7.6.18) chiziqlar oilasiga kirmasin, ya'ni (7.6.15) differentensial tenglamaning xarakteristikalaridan iborat bo'lmasin. Boshqacha aytganda, $x = \varphi_1(t)$, $y = \varphi_2(t)$ (7.6.17) tenglamaning integral egri chizig'idan iborat bo'lmasin. Bu chiziqning tenglamalarini (7.6.18) tenglikka qo'yib,

$$\psi(\varphi_1(t), \varphi_2(t)) = \bar{\psi}(t) = \bar{C} \quad (7.6.19)$$

munosabatni topamiz. Oxirgi (7.6.19) tenglikdan t o'zgaruvchini aniqlaymiz:

$$t = \omega(\bar{\psi}).$$

Endi ushbu

$$u = \varphi(\omega(\bar{\psi})) = f(\psi(x, y))$$

funksiyani qaraylik. Bu esa (7.6.15)-(7.6.16) masalaning yechimini beradi. Chunki,

$$\varphi(\omega(\bar{\psi})) = \varphi(\omega(\psi(\varphi_1(t), \varphi_2(t)))) = \varphi(\omega(\bar{\psi})) \equiv \varphi(t).$$

Faraz qilaylik, $x = \varphi_1(t)$, $y = \varphi_2(t)$ egri chiziq (7.6.17) oddiy differentensial tenglamalar sistemasining integral chizig'i bo'lsin, ya'ni

$$\psi(\varphi_1(t), \varphi_2(t)) = C = C_0 = const.$$

U holda (7.6.15) differentensial tenglamaning ixtiyoriy yechimi

$$u(x, y) = f(\psi(x, y)), \quad \forall f \in C^1$$

ko'rinishda bo'lgani uchun, u (7.6.17) oddiy differentensial tenglamaning integral chizig'i ustida o'zgarmasdir:

$$f(\psi(\varphi_1(t), \varphi_2(t))) = f(C_0) = u_0 = const. \quad (7.6.20)$$

Berilgan (7.6.15)-(7.6.16) masala yechimga ega bo'lishi uchun $u = \varphi(t) \equiv u_0$ tenglikning bajarilishi zarur va yetarli. Shuning uchun (7.6.20) shartni qanoatlantiruvchi ixtiyoriy

$$u(x, y) = f(\psi(x, y))$$

funksiya uning yechimidan iborat bo'ladi. Bu yerda u_0 ixtiyoriy tanlangan son. Bundan ko'rinadiki, bu holda (7.6.15)-(7.6.16) masala cheksiz ko'p yechimga ega bo'ladi.

7. Ushbu

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0$$

tenglamaning quyidagi

- 1) $x = 2t, y = t, u = t, t \in R$; 2) $x = t, y = t, u = t^2, t \in R$;
 3) $x = t, y = t, u = 2, t \in R$ egri chiziqlarni o'z ichiga oladigan yechimini toping, ya'ni 1), 2), 3) egri chiziqlardan o'tuvchi yechimi topilsin.

Yechish. 1) Avvalo berilgan xususiy hosilali differensial tenglamaga mos keluvchi xarakteristik tenglama tuzib olamiz:

$$\frac{dx}{y} = \frac{dy}{x}.$$

Bu tenglamani integrallab,

$$x^2 - y^2 = C = const$$

uning integralini topamiz:

$$\psi(x, y) = x^2 - y^2 = C.$$

Quyidagi

$$\psi(\varphi_1(t), \varphi_2(t)) = (2t)^2 - t^2 = 3t^2 = \bar{\psi}$$

tenglikni tuzib olamiz. Bundan

$$t = \pm \frac{1}{\sqrt{3}} \sqrt{\bar{\psi}}$$

munosabatni topamiz. Berilgan Koshi masalasining yechimi

$$u(x, y) = t = \begin{cases} \frac{\sqrt{x^2 - y^2}}{\sqrt{3}}, & t \geq 0, \\ -\frac{\sqrt{x^2 - y^2}}{\sqrt{3}}, & t < 0 \end{cases}$$

ko'rinishni oladi. Chunki

$$\bar{\psi} = \psi(\varphi_1(t), \varphi_2(t)) = \psi(x, y) = x^2 - y^2.$$

2) Qaralayotgan holda $x = t, y = t$ chiziqlar

$$\frac{dx}{y} = \frac{dy}{x}$$

tenglamaning integral chiziqlaridan iborat bo‘ladi. Bu holda (7.6.20) tenglikka asosan berilgan Koshi masalasi yechimga ega bo‘lmaydi. Chunki, $u = \varphi(t) = t^2$, bu esa o‘zgarmas emas.

3) Bu holda ham $x = t, y = t$ chiziq

$$\frac{dx}{y} = \frac{dy}{x}$$

oddiy differensial tenglamaning integral chizig‘i bo‘ladi. Lekin bu holda (7.6.20) tenglik bajariladi, ya‘ni

$$u = 2 = f(\psi(x, y)) = f(x^2 - y^2) \Big|_{\substack{x=t \\ y=t}} = f(0),$$

$$u(x, y) = f(x^2 - y^2), f(0) = 2, \forall f \in C^1$$

qaralayotgan masalaning yechimi bor.

II. Quyidagi

$$\sum_{i=1}^n a_i(x) \frac{\partial u}{\partial x_i} + b(x)u = f(x) \quad (7.6.21)$$

chizikli xususiy hosilali differensial tenglamani qaraylik. Bu yerda $x = (x_1, x_2, \dots, x_n) \in R^n$, $a_i(x), i = \overline{1, n}$, $b(x)$ va $f(x)$ funksiyalar biror $D \subset R^n$ sohada aniqlangan va uzluksiz differensiallanuvchi deb hisoblaymiz.

R^n fazoda γ -silliq gipersirt quyidagi

$$x_i = \varphi_i(s_1, s_2, \dots, s_{n-1}), i = \overline{1, n}$$

$$M(s_1, s_2, \dots, s_{n-1}) \in V$$

tenglamalar orqali berilgan bo‘lsin. Bunda $V \subset R_s^{n-1}$ soha bo‘lib, $\varphi_i(s) \in C^1(V)$ va

$$\text{rang} \begin{pmatrix} \frac{\partial \varphi_1}{\partial s_1} & \frac{\partial \varphi_1}{\partial s_2} & \dots & \frac{\partial \varphi_1}{\partial s_{n-1}} \\ \dots & \dots & \dots & \dots \\ \frac{\partial \varphi_n}{\partial s_1} & \frac{\partial \varphi_n}{\partial s_2} & \dots & \frac{\partial \varphi_n}{\partial s_{n-1}} \end{pmatrix} = n-1, s \in V.$$

shartlarni qanoatlantiradi.

Endi, (7.6.21) tenglamaning

$$u|_{\gamma} = h(s) \quad (7.6.24)$$

shartni qanoatlantiruvchi yechimini toppish bilan shug‘ullanamiz. Berilgan (7.6.21) tenglamaning xarakteristik tenglamasi ushbu

$$\frac{dx}{dt} = a(x)$$

ko‘rinishda bo‘ladi. Bu yerda $a_i(x), b(x), f(x) \in C^1(D), D \subset R^n - \gamma$ gipersirtni saqllovchi soha va $\forall x \in D$ deb qaraladi.

7.6.2-teorema. Aytaylik, γ gipersirt xarakteristkalariga urinmasin. U holda (7.6.21), (7.6.22) Koshi masalasi γ gipersirtning biror atrofida yagona yechimga ega bo‘ladi.

Isbot. Agar $l: x = x(t)$ xarakteristika bo‘lsa, h holda l ustida ushbu

$$\frac{du}{dt} = \sum_{i=1}^n \frac{\partial u}{\partial x_i} \cdot \frac{dx_i}{dt} = \sum_{i=1}^n a_i(x) \frac{\partial u}{\partial x_i} = f(x) - b(x)u$$

munosabat bajariladi. Endi γ gipersirtidan barcha xarakteristkalarini chiqarsak, u holda l_s xarakteristkalar ustida quyidagi

$$\frac{du}{dt} + b(x)u = f, u|_{t=0} = h(s)$$

Koshi masalasiga ega bo‘lamiz. $d = n - 1$, ya‘ni s_1, s_2, \dots, s_{n-1} parametrga bog‘liq bo‘lgan bu Koshi masalalari oilasini yechib $u = w(t, s)$, $s = (s_1, s_2, \dots, s_{n-1})$ silliq funksiya (yechim)ni topamiz.

8. Quyidagi

$$y \frac{\partial u}{\partial x} - x \frac{\partial u}{\partial y} = 0$$

tenglamaning yechimini toping.

Yechish. Berilgan xususiy hosilali tenglamaga mos xarakteristik tenglama tuzamiz:

$$\frac{dx}{y} = -\frac{dy}{x} = \frac{du}{0} = dt.$$

Bu munosabatni sistema ko‘rinishida yozish mumkin:

$$\begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = -x, \\ \frac{du}{dt} = 0. \end{cases}$$

Bu sistemaning birinchisini x ga, ikkinchisini y ga ko‘paytirib qo‘shamiz, natijada ushbu

$$\begin{cases} x \frac{dx}{dt} + y \frac{dy}{dt} = 0, \\ \frac{du}{dt} = 0 \end{cases}$$

sistemaga ega bo‘lamiz. Bu sistemani integrallab

$$\begin{cases} \frac{d}{dt}(x^2 + y^2) = 0, \\ \frac{du}{dt} = 0, \\ \int \frac{d}{dt}(x^2 + y^2) = C_1, \\ \int \frac{du}{dt} = C_2, \\ \begin{cases} x^2 + y^2 = C_1, \\ u = C_2. \end{cases} \end{cases}$$

Berilgan xususiy hosilali tenglamaning umumiy yechimi $F(C_1, C_2) = 0$ yoki $F(x^2 + y^2, u) = 0$ ko'rinishda bo'ladi.

9. Quyidagi

$$u_t + cu_x = 0, \quad c = \text{const}$$

tenglamani yechimini toping.

Yechish. Xarakteristik tenglamani tuzib olamiz:

$$\frac{dt}{1} = \frac{dx}{c} = \frac{du}{0} = dy$$

yoki

$$\begin{cases} \frac{du}{dy} = 0, \\ \frac{dx}{dy} = c, \\ \frac{dt}{dy} = 1. \end{cases}$$

Bu tenglamaning ikkinchi tenglamasidan, uchinchi tenglamasini c ga ko'paytirib ayiramiz:

$$\begin{cases} \frac{du}{dy} = 0, \\ \frac{dx}{dy} - c \frac{dt}{dy} = 0, \\ \frac{du}{dy} = 0, \\ \frac{d}{dy}(x - ct) = 0. \end{cases}$$

oxirgi sistemani integrallab, ushbu

$$\begin{cases} u = C_1, \\ x - ct = C_2 \end{cases}$$

munosabatni topamiz.

Natijada, berilgan xususiy hosilali tenglamaning yechimini topamiz:

$$F(C_1, C_2) = 0, \quad F(u, x - ct) = 0.$$

10. Quyidagi

$$u_t + uu_x = 0$$

Xopfa tenglamasini yechimini toping.

Yechish. Xarakteristik tenglama tuzib olamiz:

$$\frac{dt}{1} = \frac{dx}{u} = \frac{du}{0} = dy$$

yoki

$$\begin{cases} \frac{du}{dy} = 0, \\ \frac{dx}{dy} = u, \\ \frac{dt}{dy} = 1. \end{cases}$$

Bu sistemaning uchinchi tenglamasini u ga ko'paytirib, ikkinchi tenglamani ayirsak, ushbu

$$\begin{cases} \frac{du}{dy} = 0, \\ u \frac{dt}{dy} - \frac{dx}{dy} = 0 \end{cases}$$

hosil bo'ladi. Buni quyidagicha yozish mumkin:

$$\begin{cases} \frac{du}{dy} = 0, \\ \frac{d}{dy}(ut - x) = 0. \end{cases}$$

Hosil bo'lgan bu tenglamalarni integrallab

$$\begin{cases} u = C_1, \\ ut - x = C_2 \end{cases}$$

munosabatni topamiz. Bundan foydalanib, berilgan xususiy hosilali differensial tenglamaning umumiy yechimini topamiz:

$$F(C_1, C_2) = 0 \text{ yoki } F(u, ut - x) = 0.$$

Qaralayotgan holda $u(x,t)$ yechimni ushbu

$$u = f(ut - x)$$

ko'rinishda yozish mumkin. Bu yerda f -ixtiyoriy uzluksiz differensiallanuvchi funksiya.

11. Ushbu

$$(x-a)\frac{\partial u}{\partial x} + (y-b)\frac{\partial u}{\partial y} = u - c$$

tenglamaning umumiy yechimini toping. Bu yerda a, b, c -biror berilgan o'zgarmas sonlar.

Yechish. Odatdagidek, xarakteristik tenglama tuzib olamiz:

$$\frac{dx}{x-a} = \frac{dy}{y-b} = \frac{du}{u-c} = dt,$$

ya'ni

$$\begin{cases} \frac{dx}{dt} = x - a, \\ \frac{dy}{dt} = y - b, \\ \frac{du}{dt} = u - c. \end{cases}$$

Oxirgi sistemani integrallab

$$\begin{cases} x = C_1 e^t + a, \\ y = C_2 e^t + b, \\ u = C_3 e^t + c \end{cases}$$

munosabatlarni topamiz. Bu sistemaning birinchi tenglamasidan

$$e^t = \frac{x-a}{C_1}$$

topib, uni sistemaning ikkinchi va uchinchi tenglamalariga qo'yib, y va u larning qiymatlarini aniqlaymiz:

$$y = \frac{C_2}{C_1}(x-a) + b, \quad u = \frac{C_3}{C_1}(x-a) + c.$$

Bu tengliklardan

$$\frac{y-b}{x-a} = \bar{C}_1, \quad \frac{u-c}{x-a} = \bar{C}_2$$

munosabatlarni topamiz. Bu yerda

$$\bar{C}_1 = \frac{C_2}{C_1}, \quad \bar{C}_2 = \frac{C_3}{C_1}.$$

Endi, berilgan differensial tenglamaning umumiy integralini aniqlaymiz:

$$F(\bar{C}_1, \bar{C}_2) = 0, F\left(\frac{y-b}{x-a}, \frac{u-c}{x-a}\right) = 0.$$

12. Ushbu

$$u_t + 2u_x + 3u = 0,$$

$$u|_{t=0} = x^2$$

Koshi masalasi yechimini toping.

Yechish. Xarakteristik tenglama tuzamiz:

$$\begin{cases} \frac{dt}{d\tau} = 1, & \frac{dx}{d\tau} = 2, & \frac{du}{d\tau} = -3u, \\ t|_{\tau=0} = 0; & x|_{\tau=0} = s; & u|_{\tau=0} = s^2. \end{cases}$$

Bu tenglamalarni integrallab

$$t = \tau + C_1, \quad x = 2\tau + C_2, \quad u = C_3 e^{-3\tau}$$

o'zgaruvchilarni topamiz. C_1, C_2, C_3 - o'zgaruvchilarning qiymatlarini boshlang'ich shartlardan foydalanib, topamiz:

$$C_1 = 0, C_2 = s, C_3 = s^2.$$

Natijada ushbu

$$t = \tau, \quad x = 2\tau + s, \quad u = s^2 e^{-3\tau}$$

munosabatlarga ega bo'lamiz. Bu tengliklarning birinchi va ikkinchi tengliklaridan quyidagi

$$s = x - 2t, \quad t = \tau$$

o'zgaruvchilarni aniqlab, so'ngra $u(x, t)$ funksiyaning qiymatini topamiz:

$$u(x, t) = (x - 2t)^2 e^{-3t}$$

13. Ushbu

$$2u_t = u_x + xu,$$

$$u|_{t=0} = 1$$

Koshi masalasining yechimini topamiz.

Yechish. Xarakteristik tenglamalar sistemasini tuzamiz:

$$\begin{cases} \frac{dt}{d\tau} = 2, & \frac{dx}{d\tau} = -1, & \frac{du}{d\tau} = xu, \\ t|_{\tau=0} = 0; & x|_{\tau=0} = s; & u|_{\tau=0} = 1. \end{cases}$$

Bu tenglamalarni integrallab, t va x o'zgaruvchilarni topamiz:

$$\begin{cases} t = 2\tau + C_1, & x = -\tau + C_2, \\ t|_{\tau=0} = 0; & x|_{\tau=0} = s. \end{cases}$$

boshlang'ich shartlardan foydalanib

$$t = 2\tau, \quad x = -\tau + s$$

o'zgaruvchilarni aniqlaymiz. Topilgan x o'zgaruvchining qiymatini xarakteristik tenglamalar sistemasining uchinchi tenglamasiga qo'yib

$$\begin{cases} \frac{du}{d\tau} = (-\tau + s)u, \\ u|_{\tau=0} = 1 \end{cases}$$

masalani hosil qilamiz. Bundan

$$u = e^{-\tau^2/2 + s\tau}$$

Aniqlaymiz. Natijada

$$u(x, t) = e^{\frac{t^2}{8} + \frac{t}{2}x}$$

yechimni topamiz.

14. Quyidagi

$$\begin{aligned} u_t + (1 + x^2)u_x - u &= 0, \\ u|_{t=0} &= \arctg x \end{aligned}$$

Koshi masalasining yechimini toping.

Yechish. Xarakteristik tenglamalarni tuzamiz:

$$\begin{cases} \frac{dt}{d\tau} = 1, & \begin{cases} \frac{dx}{d\tau} = 1 + x^2, \\ \frac{du}{d\tau} = u, \end{cases} \\ t|_{\tau=0} = 0; & \begin{cases} x|_{\tau=0} = s; \\ u|_{\tau=0} = \arctg s. \end{cases} \end{cases}$$

Bu tenglamalarni integrallab

$$\begin{cases} t = \tau + C_1, & \begin{cases} \arctg x = \tau + C_2, \\ u = C_3 e^\tau, \end{cases} \\ t|_{\tau=0} = 0; & \begin{cases} x|_{\tau=0} = s; \\ u|_{\tau=0} = \arctg s. \end{cases} \end{cases}$$

boshlang'ich shartlardan foydalanib, ushbu

$$t = \tau, \arctg x = \tau + \arctg s, u = (\arctg s)^2 e^\tau$$

munosabatlarni topamiz. Bu tengliklarning birinchi va ikkinchi tengliklaridan

$$\arctg s = \arctg x - t$$

ifodani topib, $u(x, t)$ ni aniqlaymiz:

$$u(x, t) = (\arctg x - t)e^t.$$

7-§. Birinchi tartibli xususiy hosilali kvazi chiziqli differensial tenglama

Quyidagi

$$a(x, y, u) \frac{\partial u}{\partial x} + b(x, y, u) \frac{\partial u}{\partial y} = f(x, y, u) \quad (7.7.1)$$

birinchi tartibli xususiy hosilali kvazi chiziqli differensial tenglamani qaraylik. Bu yerda $a(x, y, u), b(x, y, u), f(x, y, u)$ funksiyalar $R^3 \supset G$ sohada

aniqlangan, uzluksiz va differensiallanuvchi, ya'ni $C^1(G)$ fazoga qarashli bo'lib, $a^2 + b^2 > 0$, $\forall(x, y, u) \in G$ shartni qanoatlantiradi.

Berilgan (7.7.1) differensial tenglamaning $u = u(x, y)$ yechimini ushbu

$$v(x, y, u) = 0 \quad (7.7.2)$$

ko'rinishda izlaymiz. Bundan foydalanib, quyidagi xususiy hosilalarni topamiz:

$$\frac{\partial u}{\partial x} = - \frac{\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial u}},$$

$$\frac{\partial u}{\partial y} = - \frac{\frac{\partial v}{\partial y}}{\frac{\partial v}{\partial u}}. \quad (7.7.3)$$

Endi (7.7.3) tenglik orqali topilgan u'_x, u'_y xususiy hosilalarning qiymatlarini (7.7.1) tenglikka qo'yamiz:

$$a(x, y, u) \begin{bmatrix} \frac{\partial v}{\partial x} \\ -\frac{\partial x}{\partial v} \\ \frac{\partial u}{\partial u} \end{bmatrix} + b(x, y, u) \begin{bmatrix} \frac{\partial v}{\partial y} \\ \frac{\partial y}{\partial v} \\ \frac{\partial u}{\partial u} \end{bmatrix} = f(x, y, u).$$

Bundan quyidagi

$$a(x, y, u) \frac{\partial v}{\partial x} + b(x, y, u) \frac{\partial v}{\partial y} - f(x, y, u) \frac{\partial v}{\partial u} = 0 \quad (7.7.4)$$

birinchi tartibli chiziqli bir jinsli differensial tenglama kelib chiqadi.

Demak, (7.7.1) ko'rinishdagi kvazi chiziqli differensial tenglamani yechimini topish masalasi (7.7.4) ko'rinishdagi bir jinsli chiziqli tenglamani yechimini topish masalasiga keltirish mumkin ekan.

7.7.1-Teorema. Aytaylik, (7.7.2) tenglama $R^2 \supset D$ sohada x, y o'zgaruvchilarning differensiallanuvchi $u = \psi(x, y)$ funksiyani aniqlasin.

Bunda $v(x, y, u)$ (7.7.4) tenglamani va $\frac{\partial v}{\partial u} \neq 0$ shartni qanoatlantiradi. U holda $u = \psi(x, y)$ funksiya (7.7.1) differensial tenglamani yechimidan iborat bo'ladi.

Isbot. $v(x, y, u)$ funksiya (7.7.4) differensial tenglamaning yechimidan iborat bo'lgani uchun, u x, y va u o'zgaruvchilarga nisbatan uzluksiz xususiy hosilalarga ega, ya'ni $\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial v}{\partial u}$ mavjud. Teorema shartiga ko'ra,

$$\frac{\partial v}{\partial u} \neq 0$$

bo'lgani uchun oshkormas funksiya mavjudligi haqidagi teoremaga asosan $\frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial y}$ uzluksiz xususiy hosilalar mavjud bo'lib, ularning qiymati (7.7.1)

tenglamaga qo'yganda u ayniyatga aylanadi. ■

Isbotlangan bu teoremadan foydalanib (7.7.1) differensial tenglamani yechimini topish algoritmini keltirish mumkin.

Avvalo (7.7.4) differensial tenglamaga mos keluvchi oddiy differensial tenglamalar sistemasini tuzib olamiz:

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{f}. \quad (7.7.5)$$

Bu esa (7.7.1) xususiy hosilali differensial tenglamaning xarakteristik tenglamasi. Uning integrallari esa xarakteristikalar deyiladi. (7.7.5) tenglama ikkita chiziqli erkli birinchi integrallarga ega:

$$\psi_1(x, y, u) = C_1, \quad \psi_2(x, y, u) = C_2.$$

U holda (7.7.4) differensial tenglamaning umumiy yechimi

$$v(x, y, u) = f(\psi_1(x, y, u), \psi_2(x, y, u)), \quad \forall f \in C^1(G)$$

ko'rinishda bo'ladi. Bunda $v(x, y, u) = 0$ deb, quyidagi

$$f(\psi_1(x, y, u), \psi_2(x, y, u)) = 0 \quad (7.7.6)$$

tenglamani hosil qilamiz. Bu tenglamani yechib $u(x, y)$ funksiyani topish mumkin. (7.7.1) tenglamaning ixtiyoriy xususiy yechimi (7.7.6) tenglamani qanoatlantirishini ko'rsatish mumkin. Shu ma'noda (7.7.6) tenglik (7.7.1) tenglamaning umumiy integralini aniqlaydi.

Misollar yechish namunalari

1. Ushbu differensial tenglamaning yechimi topilsin:

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2xy$$

Yechish. Avvalo berilgan xususiy hosilali differensial tenglamaga mos keluvchi xarakteristik tenglama tuzib olamiz:

$$\frac{dx}{x} = \frac{dy}{y} = \frac{du}{2xy}.$$

Bu sistemaning ikkita chiziqli erkli birinchi integrallarini topamiz:

$$1) \int \frac{dx}{x} = \int \frac{dy}{y} \Rightarrow \frac{y}{x} = C_1 = \text{const.}$$

$$2) \frac{dx}{x} = \frac{du}{2xy}, \quad 2y \int dx = \int du, \quad 2yx + C_2 = u, \quad u - 2xy = C_2 = \text{const.}$$

Demak quyidagi

$$\psi_1(x, y, u) = \frac{y}{x} = C_1, \psi_2(x, y, u) = u - 2xy = C_2$$

chiziqli erkli integrallar hosil bo'ladi. Ma'lumki, bu holda berilgan xususiy hosilali differensial tenglamaning umumiy yechimi

$$f(\psi_1, \psi_2) = f\left(\frac{y}{x}, u - 2xy\right) = 0, \forall f \in C^1(G)$$

ko'rinishda bo'ladi. Bu tenglamani ikkinchi argumentga nisbatan yechib,

$$u(x, y) = 2xy + \varphi\left(\frac{y}{x}\right), \forall \varphi \in C^1$$

berilgan differensial tenglamaning yechimini topamiz.

2. Ushbu Koshi masalasi yechimini toping:

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = u - y^2,$$

$$u(x, y)|_{y=2} = x - x^2.$$

Yechish. Oldingi misoldagi kabi berilgan differensial tenglamaning xarakteristik tenglamasini tuzib olamiz:

$$\frac{dx}{x} = \frac{dy}{y} = \frac{du}{u - y^2}.$$

Endi bu tenglamaning chiziqli erkli integrallarini aniqlaymiz:

$$1) \frac{dx}{x} = \frac{dy}{y} \Rightarrow \frac{x}{y} = C_1,$$

$$2) \frac{dy}{y} = \frac{du}{u - y^2} \Rightarrow \frac{u - y^2}{y} = C_2.$$

Natijada ushbu

$$\psi_1(x, y, u) = \frac{x}{y} = C_1, \psi_2(x, y, u) = \frac{u - y^2}{y} = C_2$$

chiziqli erkli integrallarni topamiz. Bundan foydalanib, $u(x, y)$ - yechim uchun

$$f(\psi_1, \psi_2) = f\left(\frac{x}{y}, \frac{u - y^2}{y}\right) = 0$$

tenglamani hosil qilamiz. Bu tenglamani ikkinchi argumentga nisbatan yechib

$$\frac{u - y^2}{y} = \varphi\left(\frac{x}{y}\right), \forall \varphi \in C^1(G),$$

$$u(x, y) = y\varphi\left(\frac{x}{y}\right) - y^2$$

umumiy yechimni hosil qilamiz.

Boshlang'ich shartdan foydalanib, quyidagi munosabatlarga ega bo'lamiz:

$$u(x, 2) = x - x^2 = 2\varphi\left(\frac{x}{2}\right) - 4,$$

$$2\varphi\left(\frac{x}{2}\right) = x - x^2 + 4,$$

$$\varphi(x) = -2x^2 + x + 2.$$

Shunday qilib quyidagi

$$u(x, y) = y\left[\frac{x}{y} - 2\left(\frac{x}{y}\right)^2 + 2\right] - y^2x - 2\frac{x^2}{y} + 2y - y^2,$$

ya'ni

$$u(x, y) = 2y - y^2 - \frac{2x^2}{y} + x$$

funksiya berilgan Koshi masalasining yechimi bo'lar ekan.

Mustaqil yechish uchun mashqlar [13], §17, №1-15; [21], §20, №946-976.

Foydalanilgan adabiyotlar

1. Agarwal R.P., O'Regan D. An introduction to ordinary differential equations. Springer. – 2000. 322 pp.
2. Айнс Э.Л. Обыкновенные дифференциальные уравнения, ГИТТЛ, 1939.
3. Арнольд В.И. Обыкновенные дифференциальные уравнения. М.: Наука, 1984, 240 с.
4. Бибииков Ю.И. Курс обыкновенных дифференциальных уравнений. М.: Высшая школа, 1991, 303 с.
5. Бойқўзиев Қ.Б. Дифференциал тенламалар. – Тошкент: Ўқитувчи, 1983. – 192 б.
6. Еругин Н.П и др. Курс обыкновенных дифференциальных уравнений. Киев: Вища школа, 1974, 287 с.
7. Коддингтон Э.А., Левинсон Н. Теория обыкновенных дифференциальных уравнений. ИИЛ, М.: 1958, 475 с.
8. Краснов М.Л., Киселев А.И., Макаренко Г.И. Сборник задач по обыкновенным дифференциальным уравнениям. М.: Высшая школа, 1978, 287 с.
9. Muxtarov Ya., A. Soleyev. Differensial tenglamalar bo'yicha misol va masalalar. Samarqand – 2014, 157 bet.
10. Найфе А. Методы возмущений. М., «Мир», 1976.
11. Петровский И.Г. Лекции по теории обыкновенных дифференциальных уравнений. М.: Изд-во МГУ, 1984, 296 с.
12. Понтрягин Л.С. Обыкновенные дифференциальные уравнения. М.: Наука, 1974.
13. Романко В.К. Курс дифференциальных уравнений и вариационного исчисления. М.: -2001, 344 с.
14. Романко В.К. Сборник задач по дифференциальным уравнениям и вариационному исчислению. М.: -2002, 255 с.
15. Самойленко А.М., Кривошениа С.А., Перестюк Н.А. Дифференциальные уравнения. Примеры и задачи. Киев: Вища школа, 1984, 408 с.
16. Салахитдинов М.С., Насритдинов Г.Н. Одний дифференциал тенгламалар. Тошкент: Ўқитувчи, 1982, 446 б.
17. Степанов В.В. Курс дифференциальных уравнений. М.:/URSS. 2006. 472 с.
18. Тихонов А.Н., Васильева А.Б., Свешников А.Г. Дифференциальные уравнения. М.: Наука, 1985.
19. Трикоми Ф. Дифференциальные уравнения. М.: 1962.
20. Yosida K. Lectures on differential and equations. New York – London – Sydney. 1960, 220 pp.

21. Федорюк М.В. Обыкновенные дифференциальные уравнения. М.: 2009, 240 с.
22. Филиппов А.Ф. Сборник задач по дифференциальным уравнениям. М.: 1970, 95 с.
23. Филиппов А.Ф. Введение в теорию дифференциальных уравнений. М.: 2010, 240 с.
24. Шарипов Ш.Р., Мўминов Н.С. Оддий дифференциал тенгламалар. Тошкент. “Ўқитувчи” 1992, 310 б.
25. Эльсгольц Л.Э. Дифференциальные уравнения и вариационное исчисление. М.: Наука, 1969, 424 с.
26. Ўринов А.Қ. Оддий дифференциал тенгламалар учун чегаравий масалалар. – Тошкент: Mumtoz so‘z, 2014. 164 б.
27. Оппоқов Ю.П, Турғунов Н., Гаффаров И.А. Оддий дифференциал тенгламалардан мисол ва масалалар тўплами. (Ўқув қўлланма). Тошкент – 2009 йил.
28. Morris Tenebout, Harry Pollard. Ordinary differential equations. Birkhhuzer. Germany, 2010.
29. Robinson J.C. An Introduction to ordinary differential equations. Cambridge University Press 2013.
30. Hasanov A.B. Shturm – Liuvill chegaraviy masalalari nazariyasiga kirish. 1 – qism. T.: Fan, 2018.
31. O‘rinov A.Q., Mirzakarimov E.M. Oddiy differensial tenglamalar Maple tizimida. Toshkent – 2013 y. 200 b.

MUNDARIJA

SO‘Z BOSHI.....	3
Oddiy differensial tenglamalar nazariyasiga kirish.....	4
I BOB. BIRINCHI TARTIBLI DIFFERENSIAL TENGLAMALAR.	8
1-§. O‘zgaruvchilari ajraladigan differensial tenglamalar.....	8
2-§. O‘zgaruvchilari ajraladigan differensial tenglamaga keltiriladigan differensial tenglamalar.....	10
3-§. Bir jinsli va kvazi bir jinsli differensial tenglamalar.....	10
4-§. Bir jinsli differensial tenglamaga keltiriladigan differensial tenglamalar.....	13
5-§. Birinchi tartibli chiziqli differensial tenglama.....	14
6-§. Noma’lum koeffitsiyentlar usuli.....	19
7-§. Bernulli differensial tenglamasi.....	21
8-§. Rikkati differensial tenglamasi.....	21
9-§. Rikkati tenglamasining maxsus ko‘rinishi.....	26
10-§. Rikkati va ikkinchi tartibli chiziqli differensial tenglama orasidagi bog‘lanishlar.....	27
11-§. To‘liq differensialli tenglamalar.....	29
12-§. Integrallovchi ko‘paytuvchi.....	31
13-§. Koshi masalasi yechimining mavjudligi va yagonaligi.....	36
14-§. Koshi masalasining korrektiligi.....	47
15-§. Differensial tenglama yechimining silliqiligi.....	50
16-§. Differensial tenglama yechimining parametrlarga va boshlang‘ich shartlarga bog‘liqligi.....	51
17-§. Kichik parametrlar usuli.....	62
18-§. Hosilaga nisbatan yechilmagan sodda differensial tenglamalar.....	71
19-§. Lagranj differensial tenglamasi.....	73
20-§. Klero differensial tenglamasi.....	74
21-§. Hosilaga nisbatan yechilmagan differensial tenglama uchun Koshi masalasi.....	75
22-§. Maxsus yechimlar va ularning mavjudligi.....	77
II BOB. YUQORI TARTIBLI DIFFERENSIAL TENGLAMALAR..	84
1-§. n-tartibli differensial tenglamalar.....	84
2-§. Ayrim n-tartibli differensial tenglamalarni yechish.....	85
3-§. n-tartibli chiziqli differensial tenglamalar.....	88
4-§. Vronskiy determinanti.....	91
5-§. n-tartibli chiziqli bir jinsli differensial tenglamaning fundamental yechimlar sistemasi.....	95
6-§. n-tartibli bir jinsli differensial tenglamani fundamental yechimlar sistemasi yordamida aniqlash.....	97
7-§. Ostrogradskiy-Liuvill formulasi.....	100

8-§. Ostragradskiy- Liuvill formulasining tatbiqi (n=2 hol).....	102
9-§. Abel formulasining umumlashmasi.....	103
10-§. n-tartibli bir jinsli o‘zgarmas koeffitsiyentli chiziqli differensial tenglama.....	106
11-§. Eyler tenglamasi.....	116
12-§. n-tartibli bir jinsli bo‘lmagan chiziqli differensial tenglama.....	118
13-§. Ayrim o‘zgarmas koeffitsiyentli chiziqli bir jinsli bo‘lmagan differensial tenglamalar.....	122
14-§. Ikkinchi tartibli chiziqli bir jinsli differensial tenglamalar.....	125
15-§. Ikkinchi tartibli chiziqli differensial tenglamaga qo‘yilgan chegaraviy masalalar.....	131
16-§. Grin funksiyasi.....	133
III BOB. DIFFERENSIAL TENGLAMANI INTEGRALLASHDA QATORLARDAN FOYDALANISH.....	141
1-§. Birinchi tartibli hosilaga nisbatan yechilgan differensial tenglamaning golomorf yechimi.....	141
2-§. Birinchi tartibli chiziqli differensial tenglamaning golomorf yechimi.....	145
3-§. Ikkinchi tartibli bir jinsli chiziqli differensial tenglamaning golomorf yechimi.....	147
4-§. Eyri tenglamasi.....	150
5-§. Bessel tenglamasi.....	152
IV BOB. DIFFERENSIAL TENGLAMALAR SISTEMASI.....	156
1-§. O‘zgarmas koeffitsiyentli chiziqli bir jinsli differensial tenglamalar sistemasi.....	156
2-§. O‘zgarmas koeffitsiyentli chiziqli bir jinsli differensial tenglamalar sistemasini karrali xos qiymatlar holda yechish algoritmi.....	160
3-§. O‘zgarmas koeffitsiyentli chiziqli bir jinsli bo‘lmagan differensial tenglamalar sistemasi.....	167
4-§. O‘zgarmas koeffitsiyentli chiziqli bir jinsli bo‘lmagan differensial tenglamalar sistemasini aniqmas koeffitsiyentlar usulida yechish.....	170
5-§. Misollar yechish na’munalari.....	172
6-§. Matritsaviy eksponenta.....	187
7-§. O‘zgarmas koeffitsiyentli chiziqli bir jinsli bo‘lmagan differensial tenglamalar sistemasini yechishda matritsaviy eksponentadan foydalanish.....	196
8-§. Differensial tenglamalar sistemasi uchun Koshi masalasi.....	197
9-§. O‘zgaruvchan koeffitsiyentli chiziqli bir jinsli bo‘lmagan differensial tenglamalar sistemasi uchun Koshi masalasi.....	203
10-§. Chiziqli bir jinsli differensial tenglamalar sistemasi. Chiziqli bog‘langan va bog‘lanmagan vektor-funksiyalar.....	209
11-§. Chiziqli bir jinsli bo‘lmagan differensial tenglamalar sistemasi...	215

V BOB. TURG‘UNLIK NAZARIYASI.....	221
1-§. Turg‘unlik tushunchasi.....	221
2-§. Chiziqli bir jinsli differensial tenglamalar sistemasi yechimining turg‘unligi.....	229
3-§. O'zgarmas koeffitsiyentli bir jinsli differensial tenglamalar sistemasi yechimining turg‘unligi.....	234
4-§. Turg‘unlikni birinchi yaqinlashish yordamida tekshirish.....	236
5-§. Turg‘unlikni Lyapunov funksiyasi yordamida tekshirish.....	240
6-§. n -tartibli chiziqli differensial tenglama yechimini turg‘unlikka tekshirish.....	248
7-§. Ko‘phadlarni turg‘unlikka tekshirish.....	251
VI BOB. AVTONOM (MUXTOR) SISTEMA VA UNING TRAYEKTORIYALARI.....	260
1-§. Muxtor sistemalar.....	260
2-§. Traektoriyalarning limitik to‘plami.....	268
3-§. Chiziqli o‘zgarmas koeffitsiyentli bir jinsli differensial tenglamalar sistemasi muvozanat (maxsus) nuqtasining klassifikatsiyasi.....	271
VII BOB. BIRINCHI INTEGRALLAR VA ULARNING TATBIQLARI.....	280
1-§. Birinchi integrallar.....	280
2-§. Gamelton tenglamalar sistemasi va Liuvill teoremasi.....	287
3-§. Harakat – burchak turidagi o‘zgaruvchi	291
4-§. Kanonik almashtirishlar.....	294
5-§. Xususiy hosilali birinchi tartibli bir jinsli differensial tenglamalar...	296
6-§. Birinchi tartibli xususiy hosilali chiziqli differensial tenglama uchun Koshi masalasi.....	301
7-§. Birinchi tartibli xususiy hosilali kvazi chiziqli differensial tenglama.....	316
Foydalanilgan adabiyotlar.....	321