O'ZBEKISTON RESPUBLIKASI XALQ TA'LIMI VAZIRLIGI

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TENGSIZLIKLAR-I.

ISBOTLASHNING KLASSIK USULLARI

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Fizika –matematika fanlari doktori, professor A. A'zamov umumiy tahriri ostida.

Qo'llanmada asosiy klassik sonli tengsizliklar hamda ularning qo'llanishiga doir turli matematik olimpiadalardagi masalalar keltirilgan.

Qo'llanma umumiy o'rta ta'lim maktablari, akademik litseylar va kasb–hunar kollejlarining iqtidorli o'quvchilari, matematika fani o'qituvchilari hamda pedagogika oliy o'quv yurtlari talabalari uchun mo'ljallangan.

Qo'llanmadan sinfdan tashqari mashg'ulotlarda, o'quvchilarni turli matematik musobaqalarga tayyorlash jarayonida foydalanish mumkin.

Taqrizchilar: TVDPI matematika kafedrasi mudiri, f.–m.f.n., dotsent Sh.B. Bekmatov
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Ushbu qo'llanma Respublika ta'lim markazi qoshidagi matematika fanidan ilmiy-metodik kengash tomonidan nashrga tavsiya etilgan. (15 iyun 2008 y., 8 -sonli bayyonnoma)

Qo'llanmaning yaratilishi Vazirlar Mahkamasi huzuridagi Fan va texnologiyalarni rivojlantirishni muvofiqlashtirish Q'omitasi tomonidan moliyalashtirilgan (ХИД 1-16 – sonli innovatsiya loyihasi**)**

§1. Sonli tengsizliklar haqida.

1. Sonli tengsizliklar va ularning xossalari.

Ta'rif: Agar $a-b$ ayirma musbat son bo'lsa, a soni b sonidan katta deyiladi va bu munosabat $a>b$ shaklida yoziladi. Agar $a-b$ ayirma manfiy bo'lsa, a soni b sonidan kichik deyiladi va $a<b$ shaklida yoziladi.

Istalgan a va b sonlar uchun quyidagi uchta munosabatdan faqat bittasi o'rinli:

1. $a-b>0 \Leftrightarrow a>b$;
2. $a-b<0 \Leftrightarrow a<b$;
3. $a-b=0 \Leftrightarrow a=b$.

Sonli tengsizliklar quyidagi xossalarga ega:

1^0. Agar $a>b$ va $b>c$ bo'lsa, $a>c$ bo'ladi (tengsizlik munosabatini tranzitivlik xossasi).

2^0. Agar $a>b$ va $c\in R$ bo'lsa, $a+c>b+c$ bo'ladi.

3^0. Agar $a>b$ va $c>0$ bo'lsa, $a\cdot c>b\cdot c$ bo'ladi.

4^0. Agar $a>b$ va $c<0$ bo'lsa, $a\cdot c<b\cdot c$ bo'ladi.

5^0. Agar $a>b$ va $c>d$ bo'lsa, $a+c>b+d$ bo'ladi.

6^0. Agar $a>b>0$ va $c>d>0$ bo'lsa, $a\cdot c>b\cdot d$ bo'ladi.

7^0. Agar $a>b>0$ va $n\in N$ bo'lsa, $a^n>b^n$ bo'ladi ($n-$toq son bo'lganda $b>0$ shart ortiqcha).

2.Tengsizliklarni isbotlashning usullari haqida.

1–misol. Istalgan a, b va c sonlari uchun $2a^2+b^2+c^2\geq 2a(b+c)$ ekanligini isbotlang.

Yechilishi. Istalgan a, b va c sonlari uchun $\left(2a^2+b^2+c^2\right)-2a(b+c)$ ayirmani manfiy emasligini ko'rsatamiz:

$$\left(2a^2+b^2+c^2\right)-2a(b+c)=\left(a^2-2ab+b^2\right)+\left(a^2-2ac+c^2\right)=$$
$$=\left(a-b\right)^2+\left(a-c\right)^2.$$

Istalgan sonning kvadrati nomanfiy son bo'lgani uchun $\left(a-b\right)^2\geq 0$ va $\left(a-c\right)^2\geq 0$. Demak, $\left(2a^2+b^2+c^2\right)-2a(b+c)$ istalgan a, b va c sonlari uchun manfiy emas. Shuning uchun berilgan tengsizlik istalgan a, b va c sonlari uchun o'rinli. Jumladan, tenglik belgisi $a=b=c$ bo'lgandagina bajariladi. Δ

Tengsizlikning to'g'riligini ko'rsatish uchun uning har ikkala qismining ayirmasini musbat yoki manfiyligini aniqlash, ya'ni 1-misoldagidek ta'rifdan foydalanib isbotlashga harakat qilish ayrim hollarda juda qiyin bo'ladi. Shuning uchun tengsizliklarni isbotlashda tengsizliklarning xossalaridan foydalaniladi.

2-misol. Musbat a, b va c sonlari uchun $\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\geq 6$ tengsizlikni isbotlang.

Yechilishi: Tengsizlikning chap qismida shakl almashtirish bajarib, uni quyidagi ko'rinishda yozamiz:

$$\left(\frac{a}{b}+\frac{b}{a}\right)+\left(\frac{a}{c}+\frac{c}{a}\right)+\left(\frac{b}{c}+\frac{c}{b}\right)\geq 6. \qquad (1)$$

Ikkita musbat son uchun o'rta arifmetik va o'rta geometrik qiymatlar orasidagi Koshi tengsizligidan foydalanamiz:

$$\frac{a}{b}+\frac{b}{a}\geq 2\sqrt{\frac{a}{b}\cdot\frac{b}{a}}=2, \quad \frac{a}{c}+\frac{c}{a}\geq 2, \quad \frac{b}{c}+\frac{c}{b}\geq 2\ .$$

Bu tengsizliklarni hadma-had qo'shib, (1) tengsizlikni hosil qilamiz.

§2. O’rtacha qiymatlar va ular orasidagi munosabatlar.

1. O’rtacha qiymatlar.

$a=\{a_1, a_2, \ldots, a_n\}$ musbat sonlar ketma-ketligi uchun

o’rta arifmetik qiymat $A(a)=A_n=\frac{a_1+a_2+\ldots+a_n}{n}$,

o’rta geometrik qiymat $G(a)=G_n=\sqrt[n]{a_1a_2\ldots a_n}$,

o’rta kvadratik qiymat $K(a)=K_n=\sqrt{\frac{a_1^2+a_2^2+\ldots+a_n^2}{n}}$ va

o’rta garmonik qiymat $N(a)=N_n=\frac{n}{a_1^{-1}+a_2^{-1}+\ldots+a_n^{-1}}$ larni aniqlaymiz.

Xususan x, y musbat sonlar uchun bu o’rta qiymatlar quyidagicha aniqlanadi:

$$A_2=\frac{x+y}{2};\quad G_2=\sqrt{xy};\quad K_2=\sqrt{\frac{x^2+y^2}{2}};\quad N_2=\frac{2xy}{x+y}.$$

2. O’rta arifmetik va o’rta geometrik qiymatlar haqida Koshi tengsizligi va uning turli isbotlari.

Teorema. $A_n \ge G_n$ va $A_n = G_n$ tenglik faqat va faqat $a_1=a_2=\ldots=a_n$ tenglik bo’lganda o’rinli.

*1-Isboti***.** $x\ge 1$ da $e^{x-1}\ge x$ ekanligi ma’lum, $e^{x-1}=x$ tenglik esa faqat $x=1$ da bajariladi. Bundan:

$$1=e^0=exp\left(\sum_{i=1}^{n}\frac{a_i}{A(a)}-1\right)=\prod_{i=1}^{n}\exp\left(\frac{a_i}{A(a)}-1\right)\ge\prod_{i=1}^{n}\frac{a_i}{A(a)}=\left(\frac{G(a)}{A(a)}\right)^n.$$

Demak, $A_n\ge G_n$ va tenglik esa faqat $\frac{a_i}{A_n(a)}=1$, $i=1, 2, \ldots, n$ bulganda bajariladi.

Bundan esa $a_1=a_2=\ldots=a_n=A_n$ ekanligi kelib chiqadi.

$A_n \ge G_n$ ekanligini isbotlaymiz: $n=2$ da $\sqrt{a_1 \cdot a_2} \le \dfrac{a_1 + a_2}{2}$. Bu tengsizlik ixtiyoriy musbat a_1 va a_2 sonlar uchun o'rinli bo'lgan $\left(\sqrt{a_1} - \sqrt{a_2}\right)^2 \ge 0$ tengsizlikdan oson hosil qilinadi. Berilgan tengsizlikni ixtiyoriy *p* ta natural sonlar uchun to'g'ri deb, *p*+1 ta natural sonlar uchun to'g'riligini isbotlaymiz. Bu sonlar $a_1, a_2, ..., a_n, a_{n+1}$ bo'lsin. a_{n+1} ularning orasida eng kattasi bo'lsin. Ya'ni, $a_{n+1} \ge a_1, ... a_{n+1} \ge a_n$. Shuning uchun $a_{n+1} \ge \dfrac{a_1 + a_2 + ... + a_n}{n}$. Quyidagicha belgilash kiritamiz:

$$A_n = \frac{a_1 + a_2 + ... + a_n}{n}, \quad A_{n+1} = \frac{a_1 + a_2 + ... + a_n + a_{n+1}}{n+1} = \frac{n \cdot A_n + a_{n+1}}{n+1}.$$

$a_{n+1} \ge A_n$ bo'lgani uchun $a_{n+1} = A_n + \alpha$ deb yozish mumkin, bu yyerda $\alpha \ge 0$. U holda $A_{n+1} = \dfrac{n \cdot A_n + A_n + \alpha}{n+1} = A_n + \dfrac{\alpha}{n+1}$. Bu tenglikni ikkala qismini (*p*+1) – darajaga ko'tarib, quyidagini topamiz:

$$\left(A_{n+1}\right)^{n+1} = \left(A_n + \frac{\alpha}{n+1}\right)^{n+1} = \left(A_n\right)^{n+1} + C_{n+1}^1 \left(A_n\right)^n \frac{\alpha}{n+1} + ... \ge$$
$$\ge \left(A_n\right)^{n+1} + \left(A_n\right)^n \cdot \alpha = \left(A_n\right)^n \cdot \left(A_n + \alpha\right) = \left(A_n\right)^n \cdot A_{n+1}.$$

Farazga ko'ra, $\left(A_n\right)^n \ge a_1 \cdot a_2 \cdot ... \cdot a_n$. Buni e'tiborga olib, $\left(A_{n+1}\right)^{n+1} \ge \left(A_n\right)^n \cdot a_{n+1} \ge a_1 \cdot a_2 \cdot ... \cdot a_n \cdot a_{n+1}$. Bundan $A_{n+1} \ge \sqrt[n+1]{a_1 \cdot a_2 \cdot ... \cdot a_n \cdot a_{n+1}}$. Tenglik $a_1 = a_2 = ... = a_n$ bo'lganda o'rinli bo'ladi.

2-isbot. Teoremaning isboti quyidagi tasdiqqa asoslangan:

Agar nomanfiy $b_1, b_2, ..., b_n$ sonlar $b_1 \cdot b_2 \cdot ... \cdot b_n = 1$ tenglikni qanoatlantirsa, u holda $b_1 + b_2 + ... + b_n \geq n$.

Bu tasdiqni masalani matematik induktsiya usulida isbotlaymiz.

$n = 1$ da masala ravshan. $n = k$ da $b_1 \cdot b_2 \cdot ... \cdot b_k = 1$ tenglikni qanoatlantiruvchi ixtiyoriy $b_1, b_2, ..., b_k$ – nomanfiy sonlar uchun $b_1 + b_2 + ... + b_k \geq k$ tengsizlik o'rinli bo'lsin. $n = k + 1$ da $b_1 \cdot b_2 \cdot ... \cdot b_{k+1} = 1$ tenglikni qanoatlantiruvchi ixtiyoriy $b_1, b_2, ..., b_{k+1}$ – nomanfiy sonlar uchun $b_1 \cdot b_2 \cdot ... \cdot b_k \cdot b_{k+1} \geq 1$ tengsizlikni qanoatlantirishini ko'rsatamiz.

Umumiylikka zarar etkazmasdan $b_k \leq 1 \leq b_{k+1}$ deb hisoblaymiz. Unda $b_1 \cdot b_2 \cdot ... \cdot b_{k-1} \cdot \left(b_k \cdot b_{k+1}\right) = 1$ bo'lgani uchun induktsiya faraziga ko'ra $b_1 + b_2 + ... + b_{k-1} + b_k \cdot b_{k+1} \geq k$ bo'ladi. Endi $b_k + b_{k+1} \geq b_k \cdot b_{k+1} - 1$ ekanligini isbotlash etarli. Bu $\left(1 + b_k\right) \cdot \left(b_{k+1} - 1\right) \geq 0$ tengsizlikka teng kuchli $b_k \leq 1 \leq b_{k+1}$ bo'lgani uchun ohirgi tengsizlik o'rinli ekanligi ravshan.

3-isbot. Teoremaning isboti quyidagi ma'lum tasdiqqa asoslangan:

$x \geq 1$ da $e^{x-1} \geq x$, shu bilan birga $e^{x-1} = x$ tenglik esa faqat $x=1$ da bajariladi. Bundan:

$$1 = e^0 = exp\left(\sum_{i=1}^{n} \frac{a_i}{A(a)} - 1\right) = \prod_{i=1}^{n} \exp\left(\frac{a_i}{A(a)} - 1\right) \geq \prod_{i=1}^{n} \frac{a_i}{A(a)} = \left(\frac{G(a)}{A(a)}\right)^n.$$

Demak, $A(a) \geq G(a)$ va tenglik esa faqat $\frac{a_i}{A(a)} = 1$, $i=1, 2, ..., n$ bo'lganda bajariladi. Bundan esa $a_1 = a_2 = ... = a_n = A(a)$ ekanligi kelib chiqadi.

1-misol. $x, y > 0$ bo'lsa, $x^2 + y^2 + 1 \geq xy + x + y$ tengsizlikni isbotlang.

Yechilishi:

$$x^2+y^2+1\ge xy+x+y\ \Rightarrow \frac{x^2}{2}+\frac{x^2}{2}+\frac{y^2}{2}+\frac{y^2}{2}+ .$$
$$+\frac{1}{2}+\frac{1}{2}=x^2+y^2+1$$

$$+\begin{cases}\dfrac{x^2}{2}+\dfrac{y^2}{2}\ge xy,\\ \dfrac{y^2}{2}+\dfrac{1}{2}\ge y,\\ \dfrac{x^2}{2}+\dfrac{1}{2}\ge x.\end{cases}\quad \Rightarrow x^2+y^2+1\ge xy+x+y.$$

2-misol. $x>0$ bo'lsa, $2^{\sqrt[12]{x}}+2^{\sqrt[4]{x}}\ge 2\cdot 2^{\sqrt[6]{x}}$ tengsizlikni isbotlang.

Yechilishi. $2^{\sqrt[12]{x}}+2^{\sqrt[4]{x}}\ge 2\cdot 2^{\sqrt[12]{x}}\cdot 2^{\sqrt[4]{x}}=2\cdot 2^{x^{\frac{1}{12}+\frac{1}{4}}}=2\cdot 2^{x^{\frac{1}{6}}}=2\cdot 2^{\sqrt[6]{x}}$.

Misollar:

1. Agar $x,\ y>0$ bo'lsa, $x^4+y^4+8\ge 8xy$ ni isbotlang.
2. $x_1,\ x_2,\ x_3, x_4, x_5>0$ bo'lsa quyidagini isbotlang:

$$x_1^2+x_2^2+x_3^2+x_4^2+x_5^2\ge x_1\left(x_2+x_3+x_4+x_5\right).$$

3. $x,y,z>0$ bo'lsa, $x^2+y^2+z^2\ge xy+yz+xz$ ni isbotlang.
4. $a,\ b,\ c>0$ bo'lsa, $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\ge 3$ ni isbotlang.
5. $a,\ b,\ c>0$ bo'lsa, $(a+1)(b+1)(c+a)(b+c)\ge 16abc$ ni isbotlang.

3. O'rta geometrik va o'rta garmonik qiymatlar orasidagi tengsizlik.

Teorema. $G(a)\ge H(a)$ ekanligini, jumladan, $H(a)=G(a)$ tenglik faqat va faqat $a_1=a_2=\ldots=a_n$ shart bajarilsa to'g'riligini isbotlang.

Isboti. Koshi tengsizligidan foydalanib (1-masalaga qarang) foydalanib

$(H(a))^{-1}= \dfrac{a_1^{-1}+a_2^{-1}+...+a_n^{-1}}{n} \geq \sqrt[n]{a_1^{-1}a_2^{-1}...a_n^{-1}} = (G(a))^{-1}$ tenglikga ega bo'lamiz. Jumladan, $H(a) = G(a)$ tenglik faqat $a_1=a_2=...=a_n$ da bajariladi.

1-misol. Agar $a, b, c>0$ bo'lsa, $\dfrac{3}{1/a+1/b+1/c} \leq \dfrac{a+b+c}{3}$ tengsizlikni isbotlang.

Yechilishi: $9 \leq (a+b+c)\left(\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\right)$,

$$\begin{cases} a+b+c \geq 3\sqrt[3]{abc}, \\ \dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c} \geq 3\cdot\dfrac{1}{\sqrt[3]{abc}}. \end{cases} \Rightarrow (a+b+c)\left(\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\right) \geq \dfrac{9\sqrt[3]{abc}}{\sqrt[3]{abc}} = 9.$$

2-misol. Agar $a, b, c>0$, $ab^2c^3=1$ bo'lsa, $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c} \geq 6$ ni isbotlang.

Yechilishi: $\dfrac{1}{a}+\dfrac{2}{b}+\dfrac{3}{c} = \dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{b}+\dfrac{1}{c}+\dfrac{1}{c}+\dfrac{1}{c} \geq 6\dfrac{1}{\sqrt[6]{ab^2c^3}} = 6$.

Misollar

1. $x, y, z>0$ bo'lsa, u holda quyidagi tengsizlikni isbotlang:

$$\frac{1}{(x+y+z)^2}+\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2} \geq \frac{28\sqrt{3}}{9\sqrt{xyz(x+y+z)}}.$$

2. Agar $x_1, x_2, ..., x_n>0$ va $x_1+x_2+...+x_n=1$ bo'lsa, u holda quyidagi tengsizlikni isbotlang:

$$\frac{x_1}{\sqrt{x_1+x_2+...+x_n}}+\frac{x_2}{\sqrt{(1+x_1)(x_2+...+x_n)}}+...+\frac{x_n}{\sqrt{(1+x_1+...+x_{n-1})x_n}} \geq 1.$$

3. $x, y, z>0$ bo'lsa, u holda quyidagi tengsizlikni isbotlang:

$$\frac{x^2-xy}{x+y}+\frac{y^2-yz}{y+z}+\frac{z^2-xz}{x+z}\ge 0.$$

4. Agar $a, b, c > 0$ bo'lsa, u holda quyidagi tengsizlikni isbotlang:

$$\sqrt{\frac{a}{b+c}}+\sqrt{\frac{b}{a+c}}+\sqrt{\frac{c}{a+b}}\ge 2.$$

5. Agar $a, b, c, d > 0$ bo'lsa, u holda quyidagi tengsizlikni isbotlang:

$$a^{p+2}+b^{p+2}+c^{p+2}\ge a^{p+2}bc+b^{p+2}ac+c^{p+2}ab.$$

4. O'rta arifmetik va o'rta kvadratik qiymatlar orasidagi tengsizlik.

Teorema. $K(a)\ge A(a)$ tengsizlik o'rinli ekanligini, jumladan, $K(a) = A(a)$ tenglik faqat $a_1=a_2=\ldots=a_n$ holdagina o'rinli bo'lishini isbotlang.

Isboti: Koshi tengsizligidan foydalanib (1-masalaga qarang) foydalanib

$2a_ia_j\le a_i^2+a_j^2,\ 1\le i<j\le n$ tengsizlikni hosil qilamiz.

Demak,

$$(a_1+a_2+\ldots+a_n)^2=a_1^2+a_2^2+\ldots+a_n^2+2\sum_{1\le i<j\le n}a_ia_j\le$$

$$\le a_1^2+a_2^2+\ldots+a_n^2+\sum_{1\le i<j\le n}(a_i^2+a_j^2)=n(a_1^2+a_2^2+\ldots+a_n^2).$$

Eslatib o'tamiz, $K(a) = A(a)$ tenglik faqat $a_1=a_2=\ldots=a_n$ o'rinli bo'ladi.

1-misol.

$H(a)\ge min\{a_1, a_2,\ldots, a_n\}$ va $max\{a_1, a_2,\ldots, a_n\}\ge K(a)$ tengsizliklarni isbotlang.

Yechimi: Umumiylikni chegaralamagan holda

$$min\{a_1, a_2,\ldots, a_n\}=a_1,\ max\{a_1, a_2,\ldots, a_n\}=a_n$$

deb hisoblash mumkin. U holda

$$H(a)=\frac{n}{a_1^{-1}+a_2^{-1}+...+a_n^{-1}}\geq\frac{n}{a_1^{-1}+a_1^{-1}+...+a_1^{-1}}=a_1\,,$$

$$K(a)=\sqrt{\frac{a_1^2+a_2^2+...+a_n^2}{n}}\leq\sqrt{\frac{a_n^2+a_n^2+...+a_n^2}{n}}=a_n \quad \text{bo'ladi.}$$

Izoh 1. yuqoridagi misollardan

$$max\{a_1, a_2, ..., a_n\} \geq K(a) \geq A(a) \geq G(a) \geq H(a) \geq min\{a_1, a_2, ..., a_n\}$$

ekanligi kelib chiqadi.

2-misol. $3\left(a^2+b^2+c^2\right)\geq\left(a+b+c\right)^2$ tengsizlikni isbotlang.

Yechilishi:

$$3a^2+3b^2+3c^2\geq a^2+b^2+c^2 2ab+2bc+2ac\Rightarrow$$

$$\Rightarrow a^2+b^2+c^2\geq ab+bc+ac\ \Rightarrow\left(a-b\right)^2+\left(b-c\right)^2+\left(c-a\right)^2\geq 0.$$

3-misol. $6\left(a^2+b^2\right)\left(a^2+b^2+c^2\right)\geq\left(a+b\right)^2\left(a+b+c\right)^2$ tengsizlikni isbotlang.

Yechilishi:

$$\times\begin{cases}2\left(a^2+b^2\right)\geq\left(a+b\right)^2\\ 3\left(a^2+b^2+c^2\right)\geq\left(a+b+c\right)^2\end{cases}\quad\Rightarrow 2a^2+2b^2\geq a^2+b^2+2ab\Rightarrow\left(a-b\right)^2\geq 0.$$

$$6\left(a^2+b^2\right)\left(a^2+b^2+c^2\right)\geq\left(a+b\right)^2\left(a+b+c\right)^2.$$

Misollar

1. Agar $a,\ b,\ c>0$ va $a+b+c=1$ bo'lsa, u holda quyidagi tengsizlikni isbotlang:

$$\sqrt{a+\left(b-c\right)^2}+\sqrt{b+\left(a-c\right)^2}+\sqrt{c+\left(a-b\right)^2}\geq\sqrt{3}\,.$$

2. Agar $a,\ b,\ c>0$ bo'lsa, u holda quyidagi tengsizlikni isbotlang:

$$\frac{a^3+b^3+c^3}{a^2b+b^2c+c^2a}+\frac{8abc}{\left(a+b\right)\left(b+c\right)\left(a+c\right)}\geq 4\,.$$

3. Agar $a, b, c \in R$ bo'lsa, u holda quyidagi tengsizlikni isbotlang:

$$(a+b)^4+(b+c)^4+(a+c)^4 \geq \frac{4}{7}\left(a^4+b^4+c^4\right).$$

4. Agar $x, y, z>0$ va $x+y+z=3$ bo'lsa, u holda quyidagi tengsizlikni isbotlang:

$$\sqrt{x}+\sqrt{y}+\sqrt{z} \geq xy+xz+yz .$$

5. Agar $a, b, c>0$ va $a^2+b^2+c^2=1$ bo'lsa, u holda quyidagi tengsizlikni isbotlang:

$$\frac{bc}{a-a^3}+\frac{ac}{b-b^3}+\frac{ab}{c-c^3} \geq \frac{5}{2}.$$

§4. Umumlashgan Koshi tengsizligi.

Teorema. $a_1, a_2, ..., a_n, p_1, p_2, ..., p_n$ – musbat sonlar bo'lsin.

$$a_1^{p_1}a_2^{p_2}...a_n^{p_n} \leq \left(\frac{a_1p_1+a_2p_2+...+a_np_n}{p_1+p_2+...+p_n}\right)^{p_1+p_2+...+p_n} \qquad (1)$$

ekanligini isbotlang, tenglik esa faqat $a_1=a_2=...=a_n$ da bajariladi.

Isboti: $s=\dfrac{a_1p_1+a_2p_2+...+a_np_n}{p_1+p_2+...+p_n}$ belgilash kiritamiz.

$e^{x-1} \geq x$ $(x \geq 1)$ tengsizlikka ko'ra $s\, e^{(a_i-1)/s} \geq a_i$, $i=1, 2, ..., n.$

Bu tengsizliklarni barchasini ko'paytirib chiqamiz:

$$a_1^{p_1}a_2^{p_2}...a_n^{p_n} \leq s^{p_1+p_2+...+p_n}\exp\left(\frac{a_1p_1+a_2p_2+...+a_np_n}{s}-p_1+p_2+...+p_n\right)=$$

$$=s^{p_1+p_2+...+p_n}.$$

Tenglik faqat $s=a_1=a_2=...=a_n$ da bajarilishi esa 1-masaladagidek isbotlanadi.

$$a_1^{p_1}a_2^{p_2}...a_n^{p_n} \le \left(\frac{a_1p_1 + a_2p_2 + ... + a_np_n}{p_1 + p_2 + ... + p_n}\right)^{p_1+p_2+...+p_n}$$

Misol. Quyidagi tengsizlikni isbotlang:

$$\left(\frac{a^3 + 4b^4 + 3c^6}{8}\right)^8 \ge a^3b^{16}c^{18}.$$

Yechilishi: Koshi tengsizligining umumiy holiga ko'ra p ning o'rnida 3 kelyapti.

1) $x \in R;\ p,q \in Q$ bo'lsa, $\sin^p x \cdot \cos^q x \le \dfrac{p^{\frac{p}{2}}q^{\frac{q}{2}}}{(p+q)^{\frac{p+q}{2}}}$ ni isbotlang.

2) $\left(\dfrac{3a^2 + 4b^3 + 5c^4}{12}\right)^{12} \ge a^6b^{12}c^{20}$

3) $a,\ b,\ c > 0$ bo'lsa, $\dfrac{a^2}{b^2} + \dfrac{b^2}{c^2} + \dfrac{c^2}{a^2} + 4\left(\dfrac{ab+bc+ac}{a^2+b^2+c^2}\right)^2 \ge 7$ isbotlang.

4) $a,\ b,\ c > 0$ bo'lsa, $\sqrt{\dfrac{a}{b} + \dfrac{b}{c} + \dfrac{c}{a}} + \sqrt{\dfrac{ab+bc+ac}{a^2+b^2+c^2}} \ge \sqrt{3} + 1$ isbotlang.

5) $a,\ b,\ c > 0$ bo'lsa, $\sqrt{\dfrac{a+b}{c} + \dfrac{b+c}{a} + \dfrac{a+c}{b}} + 2\sqrt{\dfrac{ab+bc+ac}{a^2+b^2+c^2}} \ge \sqrt{6} + 2$

isbotlang.

§5. Umumlashgan Yung tengsizligi.

Teorema.

$$a_1a_2...a_n \le \frac{a_1^{r_1}}{r_1} + \frac{a_2^{r_2}}{r_2} + ... + \frac{a_n^{r_n}}{r_n} \qquad (2)$$

tengsizlik urinli, bu yyerda $a_1, a_2, \ldots, a_n$, $r_1, r_2, \ldots, r_n$ lar musbat sonlar, jumladan, $\frac{1}{r_1}+\frac{1}{r_2}+\ldots+\frac{1}{r_n}=1$.

Isboti: 5-masaladagi (1) tenglikda a_i ni $a_i^{r_i}$ ga , r_i ni esa $1/r_i$ (i=1, 2, …, n) ga almashtirib

$$a_1 a_2 \ldots a_n \le \frac{a_1^{r_1}}{r_1}+\frac{a_2^{r_2}}{r_2}+\ldots+\frac{a_n^{r_n}}{r_n}$$ ni olamiz.

Izoh. *n=2* holida esa Yung klassik tengsizligiga ega bo'lamiz:

$$\frac{1}{p}a^p+\frac{1}{q}b^q \ge ab \quad (a\ge 0,\ b\ge 0), \tag{3}$$

bu yyerda p, q sonlar $\frac{1}{p}+\frac{1}{q}=1$ tenglikni qanoatlantiruvchi musbat sonlar.

1-misol. Agar $a, b, c > 0$ va $ab+bc+ac=abc$ bo'lsa, $abc \le \frac{a^b}{b}+\frac{b^c}{c}+\frac{c^a}{a}$

Yechilishi: Shartga ko'ra $ab+bc+ac=abc \Rightarrow \frac{1}{a}+\frac{1}{b}+\frac{1}{c}=1$

$abc \le \frac{a^b}{b}+\frac{b^c}{c}+\frac{c^a}{a}$ tengsizlik Yung tengsizligining xususiy holidan kelib chiqadi.

2-misol. Agar $a, b, c > 0$ bo'lsa, $18a^2+12b^3+6c^6 \ge 36abc$ ni isbotlang.

Yechilishi: $18a^2+12b^3+6c^6 \ge 36abc$ tengizlikni ikkala tomonini 36 ga bo'lamiz $\frac{a^2}{2}+\frac{b^3}{3}+\frac{c^6}{6} \ge abc$; $\frac{1}{r_1}+\frac{1}{r_2}+\ldots+\frac{1}{r_n}=1$ bo'lsa, Yung tengsizligi o'rinli.

$\frac{1}{2}+\frac{1}{3}+\frac{1}{6}=1 \Rightarrow \frac{a^2}{2}+\frac{b^3}{3}+\frac{c^6}{6} \ge abc$ tengsizlik o'rinli.

Misollar

1) Agar $x,\ y,\ z \geq 0$ va $x+y+z=2$ bo'lsa, $\sqrt{x^3y+y^3z+z^3x}+\sqrt{xy^3+yz^3+zx^3} \leq 2$ ni isbotlang.

2) Agar $a,\, b,\, c \geq 0$ va $a^2+b^2+c^2=3$ bo'lsa, $(2-ab)(2-bc)(2-ac) \geq 1$ ni isbotlang.

3) Agar $a,\, b,\, c \geq 0$ va $a+b+c=2$ bo'lsa, $\sqrt{a^2b+b^2c+c^2a}+\sqrt{ab^2+bc^2+ca^2} \leq 3$ ni isbotlang.

4) Agar $a,\, b,\, c \geq 0$ va $a+b+c=1$ bo'lsa, $\sqrt{a+(b-c)^2}+\sqrt{b+(c-a)^2}+\sqrt{c+(a-b)^2} \geq \sqrt{3}$ ni isbotlang.

5) Agar $a,\, b,\, c \geq 0$ bo'lsa, $\frac{a+b}{c}+\frac{b+c}{a}+\frac{a+c}{b} \geq 4\left(\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}\right)$ ni isbotlang.

§6. Gel'der tengsizligi.

Teorema. $\frac{1}{p}+\frac{1}{q}=1$ shartni qanoatlantiruvchi barcha musbat p, q sonlar va a_j, b_j, $j = 1, ..., n$ sonlar uchun

$$\left|\sum_{i=1}^{n} a_i b_i\right| \leq \left(\sum_{i=1}^{n} |a_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |b_i|^q\right)^{\frac{1}{q}} \tag{4}$$

tengsizlik har doim to'g'ri.

Isboti. $\sum_{i=1}^{n} |a_i|^p \neq 0, \sum_{i=1}^{n} |b_i|^q \neq 0$ deb faraz qilamiz (aks holda (4) tengsizlik bajarilishi ravshan). Yung tengsizligini qo'llab

$$\left|\sum_{i=1}^{n}\frac{a_i}{\left(\sum_{k=1}^{n}|a_k|^p\right)^{\frac{1}{p}}}\frac{b_i}{\left(\sum_{k=1}^{n}|b_k|^q\right)^{\frac{1}{q}}}\right| \le \left|\sum_{i=1}^{n}\frac{|a_i|}{\left(\sum_{k=1}^{n}|a_k|^p\right)^{\frac{1}{p}}}\frac{|b_i|}{\left(\sum_{k=1}^{n}|b_k|^q\right)^{\frac{1}{q}}}\right| \le$$

$$\le \left|\sum_{i=1}^{n}\frac{|a_i|^p}{p\sum_{k=1}^{n}|a_k|^p}\frac{|b_i|^q}{q\sum_{k=1}^{n}|b_k|^q}\right| = \frac{1}{p}+\frac{1}{q}=1$$

ga ega bo'lamiz. Bu yyerdan (4) tengsizlik kelib chiqadi.

*Izoh***.** Gyol'der tengsizligining $p = q = 2$ dagi $\left|\sum_{i=1}^{n}a_ib_i\right| \le \sqrt{\sum_{i=1}^{n}a_i^2}\sqrt{\sum_{i=1}^{n}b_i^2}$ *Koshi-Bunyakovskiy-Shvarts* tengsizligi deb ataluvchi bir muhim hususiy holini aytib o'tamiz.

1-misol (Minkovskiy tengsizligi). Ixtiyoriy musbat a_j, b_j $(j = 1,...,n)$ sonlar va natural p son uchun

$$\left(\sum(a_k+b_k)^p\right)^{1/p} \le \left(\sum a_k^p\right)^{1/p}\left(\sum b_k^p\right)^{1/p} \qquad (5)$$

tengsizlikni isbotlang

Yechilishi. $(a_k + b_k)^p = a_k (a_k + b_k)^{p-1} + b_k (a_k + b_k)^{p-1}$ ($k=1, 2, ..., n$) tengsizlikni qo'shib,

$$\sum (a_k + b_k)^p = \sum a_k (a_k + b_k)^{p-1} + \sum b_k (a_k + b_k)^{p-1}$$ ni olamiz.

(4) tengsizlikka ko'ra

$$\sum a_k (a_k + b_k)^{p-1} \le \left(\sum a_k^p\right)^{1/p}\left(\sum(a_k+b_k)^{q(p-1)}\right)^{1/q},$$

$$\sum b_k (a_k + b_k)^{p-1} \le \left(\sum b_k^p\right)^{1/p}\left(\sum(a_k+b_k)^{q(p-1)}\right)^{1/q}$$

larga ega bo'lamiz, bu yyerdan $q(p-1)= p$ tenglik yordamida (6) tengsizlik kelib chiqadi.

Misollar

1) $\left(a_1b_1 + a_2b_2\right) \le \left(a_1^2 + a_2^2\right)^{\frac{1}{2}} \cdot \left(b_1^2 + b_2^2\right)^{\frac{1}{2}}$ Gyol'der tengsizligining $p = q = 2$ holiga ko'ra o'rinli.

2) $\left(a_1b_1 + a_2b_2 + a_3b_3\right) \le \left(a_1^{\frac{5}{3}} + a_2^{\frac{5}{3}} + a_3^{\frac{5}{3}}\right)^{\frac{5}{2}} \cdot \left(b_1^{\frac{5}{3}} + b_2^{\frac{5}{3}} + b_3^{\frac{5}{3}}\right)^{\frac{5}{2}}$ Gyol'der tengsizligining $n = 3,\ p = \frac{5}{3},\ q = \frac{5}{2}$ holiga ko'ra o'rinli.

3) Agar $a, b, c \ge 0$ va $a^3 + b^3 + c^3 = 3$ bo'lsa, $a^4b^4 + b^4c^4 + c^4a^4 \le 3$ ni isbotlang.

4) Agar $a, b, c \ge 0$ bo'lsa, $a^2 + b^2 + c^2 + 2abc + 1 \ge 2\left(ab + bc + ac\right)$ isbotlang.

5) Agar $a, b, c \ge 0$ va $abc = 1$ bo'lsa, $\frac{a+b+c}{3} \ge \sqrt[5]{\frac{a^2 + b^2 + c^2}{3}}$ ni isbotlang.

Amaliyot uchun masalalar.

1-masala. Tengsizliklarni isbotlang:

$$n! > \left(\frac{n}{3}\right)^n, \quad n \in N; \tag{1}$$

$$n! < \left(\frac{n+1}{2}\right)^n, \quad \forall n \in N : n \ge 2; \tag{2}$$

$$n! > n^{\frac{n}{2}}, \quad \forall n \in N : n \ge 3; \tag{3}$$

$$n! > 2^{n-1}, \quad \forall n \in N : n \ge 3; \tag{4}$$

$$\left(\frac{n}{e}\right)^n < n < e\left(\frac{n}{2}\right)^n, \quad \forall n \in N. \qquad (5)$$

(1) tengsizlikni isbotlaymiz.

Induktsiya bazasi. $n=1$ da: $1! > \left(\frac{1}{3}\right)^1$ ga egamiz. Induktsiya bazasi isbotlandi.

Induktiv o'tish. $n=k$ da $k! > \left(\frac{k}{3}\right)^k$ tengsizlik to'g'ri deb faraz qilamiz. $n=k+1$ da tengsizlik bajarilishini isbotlaymiz:

$$(k+1)! > \left(\frac{(k+1)}{3}\right)^{(k+1)}. (k+1)! = k!(k+1) > \left(\frac{k}{3}\right)^k (k+1)$$

ga egamiz.

$\left(\frac{k+1}{3}\right)^{(k+1)}$ songa ko'paytiramiz va bo'lamiz:

$$\left(\frac{k+1}{3}\right)^{k+1} \frac{3^{(k+1)} \cdot k^k \cdot (k+1)}{(k+1)^{(k+1)} \cdot 3^k} = = \left(\frac{k+1}{3}\right)^{k+1} \frac{3}{(1+\frac{1}{k})^k} > \left(\frac{k+1}{3}\right)^{k+1}.$$

Bu yyerda quyidagi joriy hisoblashlarni bajaramiz:

$$(1+\frac{1}{k})^k = 1 + \frac{1}{k} + \frac{k(k-1)}{2!} \cdot \frac{1}{k^2} + \ldots + \frac{k(k-1)\cdot\ldots\cdot(k-k+1)}{k!} \cdot \frac{1}{k^k} =$$

$$= 1 + 1 + \frac{1}{2!}\underbrace{\left(1-\frac{1}{k}\right)}_{<1} + \ldots + \frac{1}{k!}\underbrace{\left(1-\frac{1}{k}\right)\left(1-\frac{2}{k}\right)\cdot\ldots\cdot\left(1-\frac{k-1}{k}\right)}_{<1} <$$

$$=1+1+\frac{1}{2!}+\underbrace{\frac{1}{3!}}_{=\frac{1}{2\cdot 3}<\frac{1}{2^2}}+\ldots+\underbrace{\frac{1}{k!}}_{=\frac{1}{1\cdot 2\cdot 3\cdot\ldots\cdot k}<\frac{1}{2^{k-1}}}<1+1+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^{k-1}}<$$

$$<1+1+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^{k-1}}+\frac{1}{2^k}+\ldots<1+\frac{1}{1-\frac{1}{2}}=3\ \Rightarrow (1+\frac{1}{k})^k<3\ \Rightarrow \frac{3}{(1+\frac{1}{k})^k}>1.$$

Matematik induktsiya printsipiga asoslanib, ixtiyoriy n natural son uchun (1) tengsizlik bajariladi deb xulosa qilamiz.

(2) tengsizlikni isbotlaymiz.

Induktsiya bazasi. $n=2$ da:

(2) tengsizlikning chap tomoni: $2!=2$;

(2) tengsizlikning o'ng tomoni: $\left(\frac{2+1}{2}\right)^2=\left(\frac{3}{2}\right)^2=\frac{9}{4}=2,25$. $2<2,25$ demak, induktsiya bazasi isbot bo'ldi.

Induktiv o'tish. $n=k$ da $k!<\left(\frac{k+1}{2}\right)^k$, $k\geq 2$ tengsizlik to'g'ri deb faraz qilamiz. $n=k+1$ da $(k+1)!<\left(\frac{k+2)}{3}\right)^{k+1}$, $k\geq 2$ tengsizlik bajarilishini isbotlash kerak. $(k+1)!=k!\cdot(k+1)<\left(\frac{k+1}{2}\right)^k\cdot(k+1)=$ ga egamiz. $\left(\frac{k+2}{2}\right)^{k+1}$ songa ko'paytiramiz va bo'lamiz:

$$=\left(\frac{k+2}{2}\right)^{k+1}\frac{(k+1)^k\cdot(k+1)\cdot 2^{k+1}}{2^k\cdot(k+2)^{k+1}}=\left(\frac{k+2}{2}\right)^{k+1}\frac{2\cdot(k+1)^{k+1}}{(k+2)^{k+1}}<\frac{2\cdot(k+1)^{k+1}}{(k+2)^{k+1}}<1$$

tengsizlik bajarilishini isbotlaymiz.

$$\frac{2\cdot(k+1)^{k+1}}{(k+2)^{k+1}}=\frac{2}{\left(\frac{k+2}{k+1}\right)^{k+1}}=2\cdot\frac{1}{\left(1+\frac{1}{k+1}\right)^{k+1}}$$

$$(1+\frac{1}{k+1})^{k+1}=1+\frac{k+1}{k+1}+\underbrace{\frac{(k+1)\cdot k}{2!}\cdot\frac{1}{(k+1)^2}+...+\left(\frac{1}{k+1}\right)^k}_{>0}>2\,.$$

$$\Rightarrow (1+\frac{1}{k+1})^{k+1}<\frac{1}{2}\;\Rightarrow 2\cdot\left(\frac{k+2}{k+1}\right)^{k+1}<2\cdot\frac{1}{2}=1\,.$$

$$<\left(\frac{k+2}{2}\right)^{k+1}\cdot 1=\left(\frac{k+2}{2}\right)^{k+1}.$$

Matematik induktsiya printsipiga asoslanib, ixtiyoriy $n\geq 2$ natural son uchun (2) tengsizlik bajariladi deb xulosa qilamiz.

(3) tengsizlikni isbotlaymiz.

Induktsiya bazasi. $n=3$ da:

(3) tengsizlikning chap tomoni: $3!=6=\sqrt{36}$;

(3) tengsizlikning o'ng tomoni: $3^{\frac{3}{2}}=\sqrt{27}$. $\sqrt{36}<\sqrt{27}$ demak, induktsiya bazasi isbot bo'ldi.

Induktiv o'tish. $n=k$ da $k!>k^{\frac{k}{2}}$, $k\geq 3$ tengsizlik to'g'ri deb faraz qilamiz.

$n=k+1$ da $(k+1)!>k^{\frac{k+1}{2}}$, $k\geq 3$ tengsizlik bajarilishini isbotlash kerak.

$$(k+1)!=k!\cdot(k+1)>k^{\frac{k}{2}}\cdot(k+1)\cdot\frac{(k+1)^{\frac{k+1}{2}}}{(k+1)^{\frac{k+1}{2}}}=(k+1)^{\frac{k+1}{2}}\cdot\frac{k^{\frac{k}{2}}\cdot(k+1)^{\frac{1}{2}}\cdot k^{\frac{1}{2}}}{(k+1)^{\frac{k}{2}}\cdot k^{\frac{1}{2}}}>$$

(6) masalada $n^{n+1} > (n+1)^n, \quad \forall n \geq 3$ tengsizlik isbotlangan edi. Bu tengsizlikdan kelib chiqadi: $n^{\frac{n+1}{2}} > (n+1)^{\frac{n}{2}}, \quad \forall n \geq 3$. U holda $n = k$ da

$$> (k+1)^{\frac{k+1}{2}} \cdot \frac{(k+1)^{\frac{k}{2}} \cdot (k+1)^{\frac{1}{2}}}{(k+1)^{\frac{k}{2}} \cdot k^{\frac{1}{2}}} = (k+1)^{\frac{k+1}{2}} \cdot \underbrace{(1+\frac{1}{k})^{\frac{1}{2}}}_{>1} > (k+1)^{\frac{k+1}{2}}.$$

Matematik induktsiya printsipiga asoslanib, ixtiyoriy $n \geq 3$ natural son uchun (3) tengsizlik bajariladi deb xulosa qilamiz.

(4) tengsizlikni isbotlaymiz.

Induktsiya bazasi. $n = 3$ da:

(4) tengsizlikning chap tomoni: $3! = 6 = \sqrt{36}$;

(4) tengsizlikning o'ng tomoni: $2^{3-1} = 4$. $6 > 4$ demak, induktsiya bazasi isbot bo'ldi.

Induktiv o'tish. (4) tengsizlik $n = k$ da bajariladi deb faraz qilamiz: $k! > 2^{k-1}, \quad \forall k \geq 3$. $n = k+1$ da $(k+1)! > 2^{k+1-1}, \quad k \geq 3$ tengsizlik bajarilishini isbotlash kerak.

$$(k+1)! = k! \cdot (k+1) > 2^{k-1} \cdot (k+1) = 2^k \cdot \underbrace{\frac{k+1}{2}}_{>1} > 2^k, \quad k \geq 3 .$$

Matematik induktsiya printsipiga asoslanib, ixtiyoriy $n \geq 3$ natural son uchun (4) tengsizlik bajariladi deb xulosa qilamiz.

(5) tengsizlikni isbotlaymiz.

Induktsiya bazasi. $n = 1$ da: $\left(\frac{1}{e}\right)^1 < 1! < e\left(\frac{1}{2}\right)^1$. Induktsiya bazasi isbot bo'ldi.

Induktiv o'tish. $n=k$ da (5) tengsizlik to'g'ri deb faraz qilamiz: $\left(\frac{k}{e}\right)^k < k! < e\left(\frac{k}{2}\right)^k$. $n=k+1$ da $\left(\frac{k+1}{e}\right)^{k+1} < (k+1)! < e\left(\frac{k+1}{2}\right)^{k+1}$ tengsizlik bajarilishini isbotlash kerak. Bu tengsizlikning chap tomonini isbotlaymiz.

$$(k+1)! = k!\cdot(k+1) > (k+1)\cdot\left(\frac{k}{e}\right)^k = \left(\frac{k+1}{e}\right)^{k+1}\cdot\frac{(k+1)\cdot\left(\frac{k}{e}\right)^k}{\left(\frac{k+1}{e}\right)^{k+1}} =$$

$$= \left(\frac{k+1}{e}\right)^{k+1}\frac{(k+1)\cdot k^k\cdot e^{k+1}}{(k+1)^{k+1}} = \left(\frac{k+1}{e}\right)^{k+1}\cdot\frac{e}{\underbrace{\left(1+\frac{1}{k}\right)^k}_{e}} > \left(\frac{k+1}{e}\right)^{k+1}.$$

(5) tengsizlikning chap tomonini isbotlaymiz.

$$(k+1)! = k!\cdot(k+1) < e\left(\frac{k}{2}\right)^k = e\left(\frac{k+1}{2}\right)^{k+1}\cdot\frac{\left(\frac{k}{2}\right)^k}{\left(\frac{k+1}{2}\right)^{k+1}} = e\left(\frac{k+1}{2}\right)^{k+1}\times$$

$$\times\underbrace{\frac{2}{(k+1)}}_{\le 1}\cdot\underbrace{\left(\frac{k}{k+1}\right)^{k+1}}_{<1} < e\left(\frac{k+1}{2}\right)^{k+1}.$$

Matematik induktsiya printsipiga asoslanib, ixtiyoriy n natural son uchun (5) tengsizlik bajariladi deb hulosa qilamiz.

Eslatib o'tamiz, (2) tengnsizlik va $\left(1+\frac{1}{n}\right)^n < e$ tengsizlikdan foydalanib, $n>1$ da

$$k! < \left(\frac{n+1}{2}\right)^n = e\cdot\left(\frac{n}{2}\right)^n \cdot \frac{\left(\frac{n+1}{2}\right)^n}{e\cdot\left(\frac{n}{2}\right)^n} = e\cdot\left(\frac{n}{2}\right)^n \cdot \frac{\left(1+\frac{1}{n}\right)^n}{e} < e\cdot\left(\frac{n}{2}\right)^n .$$

2-masala. Tengsizliklarni isbotlang:

$$x_n = \underbrace{\sqrt{2+\sqrt{2+...+\sqrt{2}}}}_{n+1 \text{ та илдиз}} < \sqrt{2}+1, \quad n\in N; \qquad (6)$$

$$x_n = \underbrace{\sqrt{4+\sqrt{4+...+\sqrt{4}}}}_{n \text{ та илдиз}} < 3, \quad n\in N. \qquad (7)$$

(6) tengsizlikni isbotlaymiz.

Induktsiya bazasi. $n=1$ da: $x_n = \sqrt{2+\sqrt{2}} < \sqrt{2+2\sqrt{2}+1} =$ $= \sqrt{(\sqrt{2}+1)^2} = \sqrt{2}+1$. Induktsiya bazasi isbotlandi.

Induktiv o'tish. $n=k$ da $x_k = \underbrace{\sqrt{2+\sqrt{2+...+\sqrt{2}}}}_{k+1 \text{ та илдиз}} < \sqrt{2}+1$ tengsizlik to'g'ri deb faraz qilamiz. $n=k+1$ da tengsizlik bajarilishini isbotlash kerak: $x_{k+1} = \underbrace{\sqrt{2+\sqrt{2+...+\sqrt{2}}}}_{k+2 \text{ та илдиз}} < \sqrt{2}+1$.

$$x_{k+1} = \underbrace{\sqrt{2+\sqrt{2+...+\sqrt{2}}}}_{k+1 \text{ та илдиз} < \sqrt{2}+1} < \sqrt{2+\sqrt{2}+1} < \sqrt{2+2\sqrt{2}+1} = \sqrt{2}+1$$

Matematik induktsiya printsipiga asoslanib, ixtiyoriy n natural son uchun (6) tengsizlik bajariladi deb xulosa qilamiz.

(7) tengsizlikni isbotlaymiz.

Induktsiya bazasi. $n=1$ da: $x_1 = \sqrt{4} < \sqrt{9}$. Induktsiya bazasi isbotlandi.

Induktiv o'tish. $n=k$ da $x_k=\underbrace{\sqrt{4+\sqrt{4+...+\sqrt{4}}}}_{k\ \text{та илдиз}}<3$ tengsizlik to'g'ri deb faraz qilamiz. $n=k+1$ da: $x_{k+1}=\sqrt{4+\underbrace{\sqrt{4+\sqrt{4+...+\sqrt{4}}}}_{k\ \text{та илдиз}\,(<3)}}<\sqrt{4+3}<3$.

Matematik induktsiya printsipiga asoslanib, ixtiyoriy n natural son uchun (7) tengsizlik bajariladi deb xulosa qilamiz.

3-masala.

$$\frac{5^n}{n!}\le\frac{5^5}{5!}\left(\frac{5}{6}\right)^{n-5},\quad \forall n\in N: n\ge 6, \tag{8}$$

tengsizlikni isbotlang.

Induktsiya bazasi. $n=6$ da: $\frac{5^6}{6!}\le\frac{5^5}{5!}\left(\frac{5}{6}\right)^{6-5}$ .Induktsiya bazasi isbotlandi.

Induktiv o'tish. $\frac{5^k}{k!}\le\frac{5^5}{5!}\left(\frac{5}{6}\right)^{k-5}$, $k\ge 6$ tengsizlik bajariladi deb faraz qilamiz. $\frac{5^{k+1}}{(k+1)!}\le\frac{5^5}{5!}\left(\frac{5}{6}\right)^{k+1-5}$ tengsizlik bajarilishini isbotlash kerak.

$$\frac{5^{k+1}}{(k+1)!}\le\underbrace{\frac{5^k}{k!}}_{\le\frac{5^5}{5!}\left(\frac{5}{6}\right)^{k-5}}\cdot\underbrace{\frac{5}{k+1}}_{<\frac{5}{6}}\le\frac{5^5}{5!}\left(\frac{5}{6}\right)^{k-5}\frac{5}{6}=\frac{5^5}{5!}\left(\frac{5}{6}\right)^{k+1-5}.$$

Matematik induktsiya printsipiga asoslanib ixtiyoriy $n\ge 6$ natural son uchun (8) tengsizlik bajariladi deb xulosa qilamiz.

4-masala. Ixtiyoriy n natural son uchun

$$|\sin kx|\le k|\sin x|, \tag{9}$$

tengsizlik o'rinli bo'lishini isbotlang.

Induktsiya bazasi. $n=1$ da: $|\sin 1x| \le 1\cdot|\sin x|$. Induktsiya bazasi isbotlandi.

Induktiv o'tish. $n=k$ da $|\sin kx| \le k|\sin x|$ tengsizlik bajariladi deb faraz qilamiz. $|\sin(k+1)x| \le (k+1)|\sin x|$ tengsizlik bajarilishini isbotlash kerak.

$$|\sin(k+1)x| = |\sin kx\cos x + \sin x\cos kx| \le \underbrace{|\sin kx|}_{\le k|\sin x|}\cdot\underbrace{|\cos x|}_{\le 1} + \sin x\underbrace{|\cos kx|}_{\le 1} \le$$

$$\le (k+1)|\sin x|.$$

Matematik induktsiya printsipiga asoslanib, ixtiyoriy n natural son uchun (9) tengsizlik bajariladi deb xulosa qilamiz.

5-masala. Ixtiyoriy n natural son uchun

$$\frac{1}{n+1}+\frac{1}{n+2}+...+\frac{1}{3n+1}>1, \qquad (10)$$

tengsizlik o'rinli bo'lishini isbotlang.

$S_k = \frac{1}{k+1}+\frac{1}{k+2}+...+\frac{1}{3k+1}$ deb belgilab olamiz.

Induktsiya bazasi. $n=1$ da: $S_1 = \frac{1}{1+1}+\frac{1}{1+2}+...+\frac{1}{3\cdot 1+1} = \frac{13}{12} > 1$.

Induktsiya bazasi isbotlandi.

Induktiv o'tish. $n=k$ da $S_k = \frac{1}{k+1}+\frac{1}{k+2}+...+\frac{1}{3k+1} > 1$ tengsizlik bajariladi deb faraz qilamiz.

$$S_{k+1} = \frac{1}{k+2}+\frac{1}{k+3}+...+\frac{1}{3k+1}+\frac{1}{3k+2}+\frac{1}{3k+3}+\frac{1}{3k+4} > 1$$

tengsizlik bajarilishini isbotlash kerak.

$$S_{k+1} = \frac{1}{k+2}+\frac{1}{k+3}+...+\frac{1}{3k+1}+\frac{1}{3k+2}+\frac{1}{3k+3}+\frac{1}{3k+4}+\left(\frac{1}{k+1}-\frac{1}{k+1}\right) =$$

$$=\underbrace{\frac{1}{k+1}+\frac{1}{k+2}+\frac{1}{k+3}+\ldots+\frac{1}{3k+1}}_{=S_1>1}+\underbrace{\frac{1}{3k+2}+\frac{1}{3k+3}+\frac{1}{3k+4}-\frac{1}{k+1}}_{>0}>1.$$

$$\frac{1}{3k+2}+\frac{1}{3k+3}+\frac{1}{3k+4}-\frac{1}{k+1}=\frac{1}{3k+2}+\frac{1}{3k+4}-\frac{2}{3k+3}=$$

$$=\frac{(3k+4)(3k+3)+(3k+2)(3k+3)-(6k+4)(3k+4)}{(3k+2)(3k+3)(3k+4)}=\frac{2}{(3k+2)(3k+3)(3k+4)}>0$$

ekanligidan ">0" tengsizlik kelib chiqadi. Matematik induktsiya printsipiga asoslanib, ixtiyoriy n natural son uchun (10) tengsizlik bajariladi deb xulosa qilamiz.

6-masala. $x^2+y^2+z^2\geq xy+xz+yz$ tengsizlikni isbotlang, bu yyerda x, y, z - musbat sonlar.

Yechilishi. Ma'lum $x^2+y^2\geq 2xy,\ x^2+z^2\geq 2xz,\ y^2+z^2\geq 2yz$ tengsizliklarni qo'shib, ushbu $(x^2+y^2)+(x^2+z^2)+(y^2+z^2)\geq 2(x^2+y^2+z^2)\geq 2(xy+xz+yz)$ tengsizlikni olamiz.

7-masala. $x^4+y^4+z^4\geq xyz(x+y+z)$ tengsizlikni isbotlang, bu yerda x, y, z - musbat sonlar.

Yechilishi. 1-masalaga ko'ra: $x^4+y^4+z^4=(x^2)^2+(y^2)^2+(z^2)^2\geq x^2y^2+y^2z^2+x^2z^2$ ga egamiz. Bu yerdan esa $x^2y^2+y^2z^2+x^2z^2\geq xyyz+yzzx+zxxy=xyz(x+y+z)$ ni olamiz.

8-masala. $x^4+y^4+z^4+u^4\geq 4xyzu$ tengsizlikni isbotlang, bu yerda x, y, z, u - musbat sonlar.

Yechilishi. $x^4+y^4\geq 2x^2y^2,\ z^4+u^4\geq 2z^2u^2$ ga egamiz. Demak, $x^4+y^4+z^4+u^4\geq 2x^2y^2+2z^2u^2$. Bundan tashqari $x^2y^2+z^2u^2\geq 2xyzu$. Demak, $x^4+y^4+z^4+u^4\geq 4xyzu$.

9-masala. $\frac{1}{2}(x+y)^2+\frac{1}{4}(x+y)\geq x\sqrt{y}+y\sqrt{x}$ tengsizlikni isbotlang, bu yerda x, y - musbat sonlar.

Yechilishi. Birinchidan, $\frac{1}{2}(x+y)^2+\frac{1}{4}(x+y)=\frac{1}{2}(x+y)(x+y+\frac{1}{2})$.

Ikkinchidan, $\frac{x+y}{2}\geq\sqrt{xy}$, $x+y+\frac{1}{2}=x+\frac{1}{4}+y+\frac{1}{4}\geq\sqrt{x}+\sqrt{y}$.

Demak, $\frac{1}{2}(x+y)^2+\frac{1}{4}(x+y)=\sqrt{xy}(\sqrt{y}+\sqrt{x})\geq x\sqrt{y}+y\sqrt{x}$.

10-masala. $x\geq 0$, $y\geq 0$ va $x+y=2$ bo'lsin.

$x^2y^2(x^2+y^2)\leq 2$ tengsizlikni isbotlang.

Aniqlik uchun $x=1+\varepsilon$, $y=1-\varepsilon$, $0\leq\varepsilon\leq 1$ deb olamiz. U holda $x^2y^2(x^2+y^2)=(1-\varepsilon)^2(1+\varepsilon)^2((1-\varepsilon)^2+(1+\varepsilon)^2)=(1-\varepsilon^2)^2(2+2\varepsilon^2)=$

$=2(1-\varepsilon^2)(1-\varepsilon^2)(1+\varepsilon^2)=2(1-\varepsilon^2)(1-\varepsilon^4)\leq 2$

11-masala. a va b bir xil ishorali sonlar bo'lsin.

$\sqrt[3]{\frac{a^2b^2(a+b)^2}{4}}\leq\frac{a^2+10ab+b^2}{12}$ ekanligini isbotlang.

Qachon tenglik bajariladi? $ab>0$ ekanligini hisobga olib va Koshi tengsizligidan foydalanib quyidagiga ega bo'lamiz:

$$\sqrt[3]{ab\cdot ab\cdot\frac{(a+b)}{2}}\leq\frac{ab+ab+\frac{a^2+2ab+b^2}{4}}{3}=\frac{a^2+10ab+b^2}{12}.$$

Tenglik esa $a=b$ bo'lganda bajarilishini eslatib o'tamiz.

12-masala. a,b va c birdan katta sonlar bo'lsin.

$\log_a(\frac{b^2}{ac}-b+ac)\log_b(\frac{c^2}{ab}-c+ab)\log_b(\frac{a^2}{bc}-a+bc)\geq 1$ tengsizlikni isbotlang.

$a>1,\ b>1,\ c>1$ va $\frac{b^2}{ac}+ac\ge 2b,\ \frac{c^2}{ab}+ab\ge 2c,\ \frac{a^2}{bc}+bc\ge 2a$ bo'lgani uchun

$$\log_a(\frac{b^2}{ac}-b+ac)\log_b(\frac{c^2}{ab}-c+ab)\log_b(\frac{a^2}{bc}-a+bc)\ge \log_a(2b-b)\times$$

$$\times\log_b(2c-c)\log_c(2a-a)=\log_a b\log_b c\log_c a=1.$$

13-masala. a va b musbat sonlar bo'lsin.

$$\sqrt[3]{\frac{a}{b}}+\sqrt[3]{\frac{b}{a}}\le\sqrt[3]{2(a+b)(\frac{1}{a}+\frac{1}{b})}$$

tengsizlikni isbotlang.

Yechilishi. Berilgan tengsizlikni kubga ko'tarish va soddalashtirishlardan so'ng quyidagiga ega bo'lamiz:

$3(\sqrt[3]{\frac{a}{b}}+\sqrt[3]{\frac{b}{a}})\le 4+\frac{a}{b}+\frac{b}{a}$. Koshi tengsizligiga ko'ra $1+1+\frac{a}{b}\ge 3\sqrt[3]{\frac{a}{b}}$ va $1+1+\frac{b}{a}\ge 3\sqrt[3]{\frac{b}{a}}$ tengsizliklarni olamiz va ularni qo'shib, qidirilayotgan tengsizlikni olamiz.

14-masala. $\frac{2n}{3n+1}\le\sum_{k=n+1}^{2n}\frac{1}{k}\le\frac{3n+1}{4(n+1)}$ $(n\in N)$ ekanligini isbotlang.

Yechilishi.

$$\sum_{k=n+1}^{2n}\frac{1}{k}=\sum_{k=n+1}^{2n}\frac{1}{3n+1-k}=\frac{1}{2}\sum_{k=n+1}^{2n}(\frac{1}{k}+\frac{1}{3n+1-k})=\frac{1}{2}\sum_{k=n+1}^{2n}\frac{1}{k}\le\frac{3n+1}{k(3n+1-k)}\ ;$$

1) $k(3n+1-k)=\frac{(3n+1)^2}{4}-(\frac{3n+1}{4}-k^2)\le\frac{(3n+1)^2}{4}$ tengsizlikdan

$\frac{1}{2}\sum_{k=n+1}^{2n}\frac{3n+1}{k(3n+1-k)}\ge\frac{4n(3n+1)}{2(3n+1)^2}=\frac{2n}{3n+1}$ ekanligi kelib chiqadi;

2) $k(3n+1-k)-2n(n+1)=(2n-k)(k-(n+1))\geq 0 \quad (n+1\leq k\leq 2n)$

tengsizlikdan $\frac{1}{2}\sum_{k=n+1}^{2n}\frac{3n+1}{k(3n+1-k)}\leq\frac{n(3n+1)}{4n(n+1)}=\frac{3n+1}{4(n+1)}$ ekanligi kelib chiqadi.

Demak $\frac{2n}{3n+1}\leq\sum_{k=n+1}^{2n}\frac{1}{k}\leq\frac{3n+1}{4(n+1)} \quad (n\in N)$.

15-masala. a, b va c musbat sonlar bo'lsin. $2(a^4+b^4+c^4)<(a^2+b^2+c^2)^2$ tengsizlik a,b va c faqat biror uchburchak tashkil qilgandagina bajarilishi mumkinligini isbotlang.

Yechilishi. Ravshanki, bizning tengsizligimiz $a^4+b^4+c^4-2a^2b^2+2b^2c^2+2a^2c^2<0$ tengsizlikka teng kuchli. Oxirgi tengsizlikning chap tomonini almashtiramiz:

$$a^4+b^4+c^4-2a^2b^2+2b^2c^2+2a^2c^2=a^2+b^2-c^2-4a^2b^2=$$
$$=(a^2+b^2-c^2-2ab)(a^2+b^2-c^2+2ab)=((a-b)^2-c^2)((a+b)^2-c^2)=$$
$$=(a-b+c)(a-b-c)(a+b+c)(a+b-c)$$

Demak, berilgan tengsizlik a, b va c biror uchburchak tashkil qilganda aniq ravishda bajariladigan ushbu $(a-b+c)(a-b-c)(a+b+c)(a+b-c)>0$ tengsizlikka teng kuchli.

Endi faraz qilamiz, bu tengsizlik bajariladi, biroq a, b va c biror uchburchak tashkil qilmaydi. U holda $a-b+c, a-b-c, a+b+c, a+b-c$ sonlardan kamida ikkitasi manfiy. $a+b-c<0$ va $b+c-a<0$ bo'lsin. Bu yerdan masalaning shartiga zid bo'lgan $2b<0$ tengsizlikni olamiz.

16-masala. $b_1, b_2, \ldots, b_n$ – $a_1, a_2, \ldots, a_n$ ketma-ketlikning biror o'rin almashtirishi bo'lsin.

$$(a_1+\frac{1}{b_1})(a_2+\frac{1}{b_2})\ldots(a_n+\frac{1}{b_n})\geq 2^n$$

tengsizlikni isbotlang.

Tengsizlikning isboti Koshi tengsizligidan darhol kelib chiqadi:

$$(a_1+\frac{1}{b_1})(a_2+\frac{1}{b_2})...(a_n+\frac{1}{b_n})\geq 2\sqrt{\frac{a_1}{b_1}}2\sqrt{\frac{a_2}{b_2}}...2\sqrt{\frac{a_n}{b_n}}=\sqrt{\frac{a_1a_2...a_n}{b_1b_2...b_n}}=2^n.$$

17-masala. $x^{10}+x^6+x^5+x^3+x^2+x+1>0$ tengsizlikni isbotlang.

Yechilishi. Ravshanki, tengsizligimiz $x\geq 0$ da bajariladi, shuning uchun x manfiy bo'lgan qiymatlarni qarash etarli. $x\leq -1$ bo'lsin, holda $x^{10}+x^5\geq 0$, $x^6+x^3\geq 0$, $x^2+x\geq 0$ va $1>0$ tengsizliklarni qo'shib, izlanayotgan tengsizlikni olamiz.

Endi $-1<x<0$ bo'lsin. Qism hollarni qaraymiz:

a) $x^5+x+1>0$. U holda

$$x^{10}+x^6+x^5+x^3+x^2+x+1=x^{10}+x^6+x^2+1+x^3+(x^5+x+1)>0.$$

b) $x^5+x+1\leq 0$. U holda

$$x^{10}+x^6+x^5+x^3+x^2+x+1=x^5(x^5+x+1)+(x^2+x^3)+(1+x)>x^5(x^5+x+1)\geq 0.$$

18-masala. $x>-1,\ y>-1$ va $z>-1$ bo'lsin.

$S=\frac{1+x^2}{1+y+z^2}+\frac{1+y^2}{1+z+x^2}+\frac{1+z^2}{1+x+y^2}\geq 2$ tengsizlikni isbotlang.

$\frac{1+x^2}{1+y+z^2}\geq\frac{1+x^2}{1+|y|+z^2}$ tengsizlik yordamida $x,\ y$ va z ning manfiy bo'lmagan qiymatlarini qarash etarli.

$$S=\frac{1+z+x^2}{1+y+z^2}+\frac{1+x+y^2}{1+z+x^2}+\frac{1+y+z^2}{1+x+y^2}-(\frac{z}{1+y+z^2}+\frac{x}{1+z+x^2}+\frac{y}{1+x+y^2})\geq$$

$$\geq 3\sqrt[3]{\frac{1+z+x^2}{1+y+z^2}\cdot\frac{1+x+y^2}{1+z+x^2}\cdot\frac{1+y+z^2}{1+x+y^2}}-(\frac{z}{1+y+z^2}+\frac{x}{1+z+x^2}+\frac{y}{1+x+y^2})\geq$$

$$\geq 3-(\frac{z}{1+y+z^2}+\frac{x}{1+z+x^2}+\frac{y}{1+x+y^2}).$$

Endi $S_1=\frac{z}{1+y+z^2}+\frac{x}{1+z+x^2}+\frac{y}{1+x+y^2}\le 1$ ekanligini isbotlaymiz. $x=0$ holni qaraymiz. U holda. Demak, $xyz=0$ da $S_1\le 1$.

$xyz\ne 0$ holda

$$S_1=\frac{1}{(x+\frac{1}{x})+\frac{z}{x}}+\frac{1}{(z+\frac{1}{z})+\frac{y}{z}}+\frac{1}{(y+\frac{1}{y})+\frac{x}{y}}\le\frac{1}{2+\frac{z}{x}}+\frac{1}{2+\frac{y}{z}}+\frac{1}{2+\frac{x}{y}}.$$

$\frac{x}{y}=a,\ \frac{y}{z}=b$ va $\frac{z}{x}=c$ deb belgilab olamiz. $abc=1$ va $S_1=\frac{1}{2+a}+\frac{1}{2+b}+\frac{1}{2+c}$ ekanligi ravshan. U holda

$$\frac{1}{2+a}+\frac{1}{2+b}+\frac{1}{2+c}=\frac{(2+b)(2+c)+(2+a)(2+c)+(2+a)(2+b)}{(2+a)(2+b)(2+c)}=$$

$$=\frac{12+4(a+b+c)+(ab+bc+ac)}{8+4(a+b+c)+2(ab+bc+ac)+abc}\le$$

$$\le\frac{12+4(a+b+c)+(ab+bc+ac)}{8+4(a+b+c)+(ab+bc+ac)+3\sqrt[3]{a^2b^2c^2}+abc}=\frac{12+4(a+b+c)+(ab+bc+ac)}{12+4(a+b+c)+(ab+bc+ac)}=1.$$

Demak, $S_1\le 1$.

19-masala. Ixtiyoriy musbat $a_j,\ b_j\ (j=1,2,...,n)$ sonlar uchun

$$\sqrt[n]{a_1...a_n}+\sqrt[n]{b_1...b_n}\le\sqrt[n]{(a_1+b_1)...(a_n+b_n)}$$

tengsizlik o'rinli ekanligini isbotlang.

Yechilishi. Gyuygens tengsizligiga asoslanib $\left(1+\frac{a_1}{b_1}\right)...\left(1+\frac{a_n}{b_n}\right)\ge\left(1+\sqrt[n]{\frac{a_1}{b_1}...\frac{a_n}{b_n}}\right)^n$

yoki $(a_1+b_1)...(a_n+b_n)\ge\left(\sqrt[n]{a_1...a_n}+\sqrt[n]{b_1...b_n}\right)^n$ ni olamiz. Bu yerdan esa $\sqrt[n]{a_1...a_n}+\sqrt[n]{b_1...b_n}\le\sqrt[n]{(a_1+b_1)...(a_n+b_n)}$ kelib chiqadi.

20-masala. Ixtiyoriy $a_1, a_2, \dots, a_n$ musbat sonlar uchun $(a_1+a_2+\dots+a_n)\left(\frac{1}{a_1}+\frac{1}{a_2}+..+\frac{1}{a_n}\right)\geq n^2$ tengsizlik o'rinli bo'lishini isbotlang.

Yechilishi. Koshi–Bunyakovskiy–Shvarts tengsizligiga ko'ra

$$n^2=\left(\sqrt{a_1}\cdot\frac{1}{\sqrt{a_1}}+\sqrt{a_2}\cdot\frac{1}{\sqrt{a_2}}+\dots+\sqrt{a_n}\cdot\frac{1}{\sqrt{a_n}}\right)^2\leq$$

$$\leq\left((\sqrt{a_1})^2+(\sqrt{a_2})^2+\dots+(\sqrt{a_n})^2\right)\sqrt{a_2}\cdot\left(\left(\frac{1}{\sqrt{a_1}}\right)^2+\left(\frac{1}{\sqrt{a_2}}\right)^2+..+\left(\frac{1}{\sqrt{a_n}}\right)^2\right)$$

ni olamiz.

21-masala. $\frac{\left(a_1+a_2+\dots+a_n\right)^2}{a_1^2+a_2^2+\dots+a_n^2}\leq\frac{a_1}{a_1+a_3}+\frac{a_2}{a_3+a_4}+\dots+\frac{a_n}{a_1+a_2}$

tengsizlikni isbotlang, bu yerda $a_k\geq 0\ (k=1,2,...,n)$.

Yechilishi. Koshi–Bunyakovskiy–Shvarts tengsizligiga ko'ra

$$(a_1+a_2+\dots+a_n)^2=\left(\sqrt{\frac{a_1}{a_1+a_3}}\cdot\sqrt{a_1(a_1+a_3)}+\dots+\sqrt{\frac{a_n}{a_1+a_2}}\cdot\sqrt{a_n(a_1+a_2)}\right)^2\leq$$

$$\leq\left(\frac{a_1}{a_1+a_3}+\frac{a_2}{a_3+a_4}+\dots+\frac{a_n}{a_1+a_2}\right)\left(a_1(a_1+a_3)+\dots a_n(a_1+a_2)\right)\leq$$

$$\geq\left(\frac{a_1}{a_1+a_3}+\frac{a_2}{a_3+a_4}+\dots+\frac{a_n}{a_1+a_2}\right)\left(\frac{1}{2}(a_1^2+a_2^2)+\frac{1}{2}(a_1^2+a_3^2)\right)+\dots+$$

$$+\left(\frac{1}{2}(a_{n-1}^2+a_n^2)+\frac{1}{2}(a_n^2+a_1^2)\right)+\left(\frac{1}{2}(a_n^2+a_1^2)+\frac{1}{2}(a_n^2+a_2^2)\right)=$$

$$=\left(\frac{a_1}{a_1+a_3}+\frac{a_2}{a_3+a_4}+\dots+\frac{a_n}{a_1+a_2}\right)(2a_1^2++\dots+2a_n^2).$$

Mashqlar

1. Agar $a, b, c, d > 0,\ c+d \le a,\ c+d \le b$, bo'lsa, u holda $ab+bc \le ab$ tengsizlik o'rinli bo'lishini isbotlang.
2. Agar x, y, z lar xaqiqiy sonlar to'plamiga tegishli bo'lsa, $x^2+y^2+z^2 \ge xy+yz+xz$ tengsizlikni isbotlang.
3. Agar $x+y+z=1$ bo'lsa, $x^2+y^2+z^2 \ge \frac{1}{3}$ ni isbotlang.
4. Agar $ab>0$ bo'lsa, $\frac{a}{b}+\frac{b}{a} \ge 2$ tengsizlikni isbotlang.
5. Xar qanday $n \ge 2$ ($n \in N$) larda $\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2}<1.$ tengsizlik o'rinli bo'lishini isbotlang.
6. Xar qanday $n \ge 2$ ($n \in N$) larda $\frac{n}{2}<1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{2^n-1}$ tengsizlik o'rinli bo'lishini isbotlang
7. $n \in N$ bo'lsa, $\frac{1}{9}+\frac{1}{25}+\ldots+\frac{1}{(2n+1)^2}$ tengsizlikni isbotlang.
8. $n \in N$ bo'lsa, $\frac{1}{2}<\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n}<\frac{3}{4}$ tengsizlikni isbotlang.
9. Agar $a_1^{\ 2}+a_2^{\ 2}+\ldots+a_n^{\ 2}=b_1^{\ 2}+b_2^{\ 2}+\ldots+b_n^{\ 2}=1$ bo'lsa, $-1 \le a_1b_1+a_2b_2+\ldots+a_nb_n \le 1$ tengsizlikni isbotlang.
10. Agar $a_1a_2\cdot\ldots\cdot a_n=1,\ a_1, a_2, \ldots a_n>0$ bo'lsa, $(1+a_1)(1+a_2)\ldots(1+a_n) \ge 2^n$ tengsizlikni isbotlang.
11. Agar $a+b \ge 1$ bo'lsa, $a^4+b^4 \ge \frac{1}{8}$ tengsizlikni isbotlang.

12. a, b musbat sonlar va birdan farqli bo'lsa, $\left|\log_a b\right| + \left|\log_b a\right| \geq 2$ tengsizlikni isbotlang.

13. $\frac{1}{\log_2 \pi} + \frac{1}{\log_\pi 2} > 2$ tengsizlikni isbotlang.

14. Agar $n \in N$ bo'lsa, $1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!} < 3$ tengsizlikni isbotlang.

15. Agar $n \in N$ bo'lsa, $2\left(\sqrt{n+1} - 1\right) < 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} < 2\sqrt{n}$ tengsizlikni isbotlang.

16. Agar $a_k = a_{k-1} + a_{k-2}$ $(k = 3, 4, \ldots)$ bo'lsa,

$\frac{1}{2} + \frac{1}{2^2} + \frac{2}{2^3} + \frac{3}{2^4} + \frac{5}{2^5} + \ldots + \frac{a_n}{2^n} < 2$ tengsizlikni isbotlang.

17. Agar $n \in N$ bo'lsa, $1 < \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{3n+1} < 2$ tengsizlikni isbotlang.

18. Agar $a_i > 0,\ i = 1, 2, \ldots, n$ bo'lsa, $\frac{a_1 + a_2 + \ldots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdot \ldots \cdot a_n}$ tengsizlikni isbotlang.

19. Agar $a_i > 0,\ i = 1, 2, \ldots, n$ bo'lsa,

$\left(a_1 + a_2 + \ldots + a_n\right)\left(\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}\right) \geq n^2$ tengsizlikni isbotlang.

20. Agar $a > 0$ bo'lsa, $\frac{1 + a + a^2 + \ldots + a^{2n}}{a + a^2 + \ldots + a^{2n-1}} \geq \frac{n+1}{n}$ tengsizlikni isbotlang.

21. Agar $0<\alpha_1<\alpha_2<\ldots<\alpha_n<\dfrac{\pi}{2}$ bo'lsa,

$tg\alpha_1<\dfrac{\sin\alpha_1+\sin\alpha_2+\ldots+\sin\alpha_n}{\cos\alpha_1+\cos\alpha_2+\ldots+\cos\alpha_n}<tg\alpha_n$ tengsizlikni isbotlang.

22. Agar $n\in N$ bo'lsa, quyidagi tengsizlikni isbotlang.

$2!\cdot 4!\cdot 6!\cdot\ldots\cdot(2n)!\geq\left((n+1)!\right)^n$

23. Agar $0<\varphi<\dfrac{\pi}{2}$ bo'lsa, $ctg\dfrac{\varphi}{2}\geq 1+ctg\varphi$ tengsizlikni isbotlang.

24. $k,l-$butun sonlar va $\alpha\neq\pm\beta+2\pi n$ bo'lsa,

$\left|\dfrac{\cos k\alpha\cos l\beta-\cos l\alpha\cos k\beta}{\cos\alpha-\cos\beta}\right|\leq\left|k^2-l^2\right|$ tengsizlikni isbotlang.

25. Agar $n>2$ bo'lsa, $(n!)^2>n^n$ tengsizlikni isbotlang.

26. Agar $a, b, p, q>0$ va p, q ratsional sonlar $\dfrac{1}{p}+\dfrac{1}{q}=1$ shartni qanoatlantirsa $ab\leq\dfrac{a^p}{p}+\dfrac{b^q}{q}$ tengsizlikni isbotlang.

27. Agar $n\in N$ bo'lsa, $2<\left(1+\dfrac{1}{n}\right)^n<3$ tengsizlikni isbotlang.

28. Agar $n>0$ bo'lsa, quyidagi tengsizlikni isbotlang. $\left(\dfrac{n}{3}\right)^n<n!$

29. Agar $n>0$ bo'lsa, quyidagi tengsizlikni isbotlang. $(3n)!>(n)^{3n}$

30. Agar $s=a_1+a_2+\ldots+a_n;\ a_i>0,\ i=1,2,\ldots,n$ bo'lsa,

$(1+a_1)(1+a_2)\cdot\ldots\cdot(1+a_n)\leq 1+\dfrac{s}{1!}+\dfrac{s^2}{2!}+\ldots+\dfrac{s^n}{n!}$ tengsizlikni isbotlang.

31. a) $\frac{a}{a^2+a+1} \le \frac{1}{3}$; b) $\frac{a}{a^2-4a+9} \le \frac{1}{2}$;

s) $a^2+\frac{1}{a^2} \ge a+\frac{1}{a}$; d) $\frac{a^4+16}{a^2+4} \ge 2a$.

32. a) $9a^2-30|a|+25 \ge 0$; b) $b^2+25 \ge 10|b|$;

s) $a^2-4a+5 \ge 2|a-2|$; d) $b^2-2b+10 \ge 6|b-1|$.

33. a) $a^4+b^4 \ge a^3b+ab^3$; b) $a^4+b^4+ab(a^2+b^2) \ge 0$;

s) $a^6+b^6 \ge a^4b^2+a^2b^4$; d) $a^6+b^6 \ge a^5b+ab^5$.

34. Agar $a \ge 0$ va $b \ge 0$ bo'lsa, u holda quyidagilarni isbotlang:

a) $a^3+b^3 \ge a^2b+ab^2$; b) $(a+b)^3 \le 4(a^3+b^3)$;

s) $a^5+b^5 \ge a^4b+ab^4$; d) $a^5+b^5 \ge a^3b^2+a^2b^3$.

35. Agar $a \ge 0$ va $b \ge 0$ bo'lsa, u holda quyidagilarni isbotlang:

a) $a^3-b^3 \ge a^2b-ab^2$; b) $a^3-b^3 \ge 3ab(a-b)$;

s) $a^3-b^3 \ge ab^2-a^2b$; d) $a^5-b^5 \ge a^4b-ab^4$.

36. a va b sonlarning ixtiyoriy qiymatlarida tengsizlik o'rinli bo'lishini isbotlang:

a) $a^4-2a^3b+2a^2b^2-2ab^3+b^4 \ge 0$;

b) $a^4-4a^3b+8a^2b^2-16ab^3+16b^4 \ge 0$.

37. Ixtiyoriy a, b, c, va d sonlar uchun tengsizlik

a) $(a^2-b^2)(c^2-d^2) \le (ac-bd)^2$;

b) $(a^2+b^2)(c^2+d^2) \ge (ac+bd)^2$.

o'rinli bo'lishini isbotlang, jumladan tenglik bajariladi shu holda va faqat shu holdaki, qachonki $ad = bc$.

38. $ab \geq 0$ shartni qanoatlantiruvchi ixtiyoriy a va b sonlar uchun $(a^2 - b^2)^2 \geq (a-b)^4$ tengsizlik o'rinli bo'lishini isbotlang.

39. Agar $a < b$ bo'lsa, $a < \frac{a+b}{2} < b$ bo'lishini isbotlang.

40. Agar $a < b < c$ bo'lsa, $a < \frac{a+b+c}{3} < b$ bo'lishini isbotlang.

41. Agar $a > 0,\ b > 0,\ c < 0,\ d < 0$ ekanligi ma'lum.

$$abc,\ bcd,\ \frac{ab}{c},\ \frac{ac}{d},\ \frac{c}{ad},\ \frac{b}{cd},\ abcd,\ \frac{ac}{bd},\ \frac{abd}{c}$$

ifodalar qanday ishoralarga ega bo'ladi ?

42. Agar

a) a va b bir xil ishorali sonlar;

b) a va b turli ishorali sonlar ekanligi ma'lum bo'lsa,

ab ko'paytma va $\frac{a}{b}$ kasr qanday ishoralarga ega bo'ladi ?

43. Agar

a) $ab > 0$; b) $\frac{a}{b} > 0$; s) $ab < 0$; d) $\frac{a}{b} < 0$; e) $a^2 b > 0$;

f) $a^2 b < 0$; g) $\frac{a}{b^2} < 0$ ekanligi ma'lum bo'lsa, a va b sonlarning ishorasini toping.

44. $a > 2$ ekanligi ma'lum bo'lsa, ifodaning ishorasi qanaqa ?

a) $3a - 6$; b) $10 - 5a$; s) $2a - 2$; d) $(a-2)(1-a)$; e) $\frac{a-2}{a-1}$;

f) $(a-3)^2(a-1)$; g) $\frac{-5}{2-a}$; h) $\frac{(a-1)(2-a)}{(5+a)}$.

45. $a < 3$ ekanligi ma'lum bo'lsa, ifodaning ishorasi qanday bo'ladi ?

a) $2a-6$; b) $12-4a$; s) $2a-8$; d) $(a-5)(a-3)$; e) $\dfrac{a-4}{3-a}$;

f) $(a-1)^2(a-2)$; g) $\dfrac{2}{3-a}$; h) $\dfrac{a-1}{(a-2)(3-a)}$.

46. Agar a) $a<1$; b) $a>4$; s) $1<a<4$; d) $a>5$ ekanligi ma'lum bo'lsa $(a-1)(a-4)$ ifoda qaysi ishoraga ega bo'ladi ?

47. Agar $a>1$ va $b>1$ bo'lsa, u holda $ab+1>a+b$ ekanligini isbotlang.

48. Agar $a>b$ va $b<2$ bo'lsa, u holda $b(a+2)>b^2+2a$ ekanligini isbotlang.

49. Agar $a>b>1$ bo'lsa, u holda $a^2b+b^2+a>ab^2+a^2+b$ ekanligini isbotlang.

50. Agar $a<b<2$ bo'lsa, u holda $a^2b+2b^2+4a<ab^2+2a^2+4b$ ekanligini isbotlang.

51. Agar $1<a<b<2$ bo'lsa, u holda $a^2b-ab^2-a^2-ab+2b^2+2a-2b>0$ ekanligini isbotlang.

52. Agar $a\geq b\geq c$ bo'lsa, u holda $a^2(b-c)+b^2(c-a)-c^2(a-b)\geq 0$ ekanligini isbotlang.

53. $\sin x > x-\dfrac{x^3}{6}$ $(x>0)$ tengsizlikni isbotlang.

54. Sonlarni taqqoslang.

a) $\dfrac{\ln 2004}{\ln 2005}$ va $\dfrac{\ln 2005}{\ln 2006}$

b) $\cos(\sin 2006)$ va $\sin(\cos 2006)$

55. $x>0$ uchun $1+2\ln x\leq x^2$ tengsizlik o'rinli bo'lishini isbotlang.

56. $x_1, x_2, \ldots, x_n$ musbat sonlar bo'lsin.

$$f(\alpha)=\begin{cases}\left(\dfrac{x_1^\alpha+\ldots+x_n^\alpha}{n}\right)^{\frac{1}{\alpha}}, & \alpha\neq 0\\ \sqrt[n]{x_1\cdot\ldots\cdot x_n}, & \alpha=0\end{cases}$$

funtsiyaning monoton o'suvchi bo'lishini isbotlang. Bundan tashqari f funktsiya qat'iy o'suvchi bo'ladi, faqat va faqat shu holdaki, qachonki x_j sonlarning hammasi o'zaro teng bo'lmasa.

57. $\sin\alpha\sin\beta\sin\gamma\leq\dfrac{3\sqrt{3}}{8}$ tengsizlikni isbotlang, bu yerda α,β va γ biror uchburchakning ichki burchaklari.

58. $a_1, a_2, \ldots, a_n$ lar $a_k\in(0:\frac{1}{2})$ $(k=1,\ldots,n)$ va $a_1+a_2+\ldots+a_n=1$ xossalarga ega bo'lgan sonlar bo'lsin. $\left(\dfrac{1}{a_1^2}-1\right)\left(\dfrac{1}{a_2^2}-1\right)\ldots\left(\dfrac{1}{a_n^2}-1\right)\geq(n^2-1)^n$ ekanligini isbotlang.

59. Ixtiyoriy a, b, c musbat sonlar uchun

$$a+b+c\leq\frac{a^2+b^2}{2c}+\frac{b^2+c^2}{2a}+\frac{a^2+c^2}{2b}\leq\frac{a^3}{bc}+\frac{b^3}{ac}+\frac{c^3}{ab}$$

tengsizlik bajarilishini isbotlang.

60. Agar $1<a\leq b\leq c$ bo'lsa, u holda

$$\frac{a}{\ln a}+\frac{b}{\ln b}+\frac{c}{\ln c}\leq\frac{1}{3}(a+b+c)\left(\frac{1}{\ln a}+\frac{1}{\ln b}+\frac{1}{\ln c}\right)\leq\frac{a}{\ln c}+\frac{b}{\ln b}+\frac{c}{\ln c}\frac{a^3}{bc}+\frac{b^3}{ac}+\frac{c^3}{ab}\quad ?$$

bo'lishini isbotlang.

61. $\sqrt[n]{2-\sqrt{3}}+\sqrt[n]{2+\sqrt{3}}\geq 2,\ \ (n=2, 3, 4,\ldots)$ ekanligini isbotlang.

62. Ixtiyoriy a, b, c nomanfiy sonlar uchun

$$(a+b)(b+c)(c+a)\geq 8abc$$

tengsizlik o'rinli bo'lishini isbotlang.

63. Ixtiyoriy $a_1, a_2, \dots, a_n$ musbat sonlar uchun

$$\sqrt{\frac{a_1+a_2}{a_3}}+\sqrt{\frac{a_2+a_3}{a_4}}+\dots+\sqrt{\frac{a_{n-1}+a_n}{a_1}}+\sqrt{\frac{a_n+a_1}{a_2}}\geq n\sqrt{2}$$

tengsizlik o'rinli bo'lishini isbotlang.

Yensen tengsizligi:

64. $\sqrt{\frac{a_1+a_2}{a_3}}+\sqrt{\left(\sum a_i\right)^2+\left(\sum b_i\right)^2}\leq\sum\sqrt{a_i^2+b_i^2}$ $(a_i, b_i > 0)$ tengsizlikni isbotlang. (Ko'rsatma: $y=\sqrt{1+x^2}$).

65. $\sum_{i=1}^{n}\frac{a_i}{S-a_i}\geq\frac{n}{n-1}$ tengsizlikni isbotlang, bu yerda $S=a_1+a_2+...a_n,\ \ a_i>0$.

66. $(\sum_{i=1}^{n}x_i)^p\geq n^{p-1}\cdot\sum_{i=1}^{n}x_i^p,\quad p>1,\ x_i>0$ tengsizlikni isbotlang.

Koshi-Bunyakovskiy tengsizligi:

67. $(a^2+b^2+c^2)(h_a^2+h_b^2+h_c^2)\geq 36S^2$ tengsizlikni isbotlang, bu yerda a, b, c lar uchburchakning tomonlari; h_a, h_b, h_c lar uchburchakning shu tomonlarga tushirilgan balandliklari; S uchburchakning yuzi.

66. $ab+\sqrt{(1-a^2)(1-b^2)}\leq 1,\ |a|\leq 1,\ |b|\leq 1$ ekanligini isbotlang**.**

68. Agar $a+2b+3c=14$ bo'lsa, $a^2+b^2+c^2\geq 14$ bo'lishini isbotlang.

Koshi tengsizligi:

69. a, b, c nomanfiy sonlar uchun $a^2+b^2+c^2=1$ shart bajariladi. $a+b+c\leq\sqrt{3}$ ekanligini isbotlang.

70. $a,b,c\geq 0,\ a+b+c=1$ berilgan. $(1-a),(1-b)(1-c)\geq 8abc,\ a+b+c=1$ tengsizlikni isbotlang.

71. Isbotlang: $abc\geq(a+b-c)(a+c-b)(b+c-a),\ (1-b)(1-c)8abc,\ a+b+c=1$

72. $x, y, z \geq 0, \ xyz = 1$ berilgan. $(3x+2y+z)(3y+2z+x)(3z+2x+y) \geq 216$ ni isbotlang.

Bernulli tengsizligi:

75. $\sqrt{1-x}+\sqrt{1+x}+\sqrt[4]{1-x^2}+\sqrt[4]{1+x^2}=4$ tenglamani yeching.

76. $\sqrt[4]{1-x}+\sqrt[4]{1+x}=4$ tenglamani yeching.

77. $\sqrt[4]{1-\frac{x}{3}}+\sqrt[6]{x+1}=\left(1-\frac{x}{24}\right)^4+\left(1+\frac{x}{36}\right)^6$ tenglamani yeching.

78. $(\frac{a}{b})^{10}+(\frac{b}{c})^{10}+(\frac{c}{d})^{10}+(\frac{d}{a})^{10} \geq abcd(\frac{1}{a^4}+\frac{1}{b^4}+\frac{1}{c^4}+\frac{1}{d^4})$ tengsizlikni isbotlang.

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